Math 333 (2015) Assignment 4

(Due: November 12, 2015)

Maximum 55 points

1. (15) In each of the following an inner product space V with its associated inner product are defined. For the indicated vectors $u, v \in V$ compute $< u, v >$ and $d(u, v) = \| u - v \|$.

a)

$$\begin{aligned} V &= \mathbb{R}^3 \quad , \quad < u, v > \equiv (Au)^T (Av) \quad , \quad A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ u &= (1, 1, 1) \quad , \quad v = (2, 0, 1) \end{aligned}$$

b)

$$\begin{aligned} V &= C[0, 1] \quad , \quad < u, v > \equiv \int_0^1 u(x)v(x)dx \quad , \\ u &= x - 1 \quad , \quad v = x + 1 \end{aligned}$$

c)

$$\begin{aligned} V &= M_{22} \quad , \quad < u, v > \equiv Tr(u^T v) \quad , \\ u &= \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad , \quad v = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

2. (5) If $V = C[0, 1]$ and $< u, v > \equiv \int_0^1 u(x)v(x)dx$, find all α (if any) for which $u = \alpha - 3x$ and $v = \alpha x + 1$ are orthogonal.

3. (15) For each of the following subspaces W of V, find a basis for the orthogonal complement $W^\perp$. Write your final answer as $W^\perp = span\{something\}$.

a) $V = \mathbb{R}^4$

$$W = row(A) \quad , \quad A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \quad < u, v > \equiv u_1 v_1 + u_2 v_2 + u_3 v_3$$

b) $V = span\{v_1, v_2, v_3, v_4, v_5\} = span\{1, \sin(x), \cos(x), \sin(2x), \cos(2x)\}$.

Note: $v_k(x)$ defining V are mutually orthogonal.

$$W = span\{\cos^2(x), \sin^2(x)\} \quad < u, v > \equiv \int_0^{2\pi} u(x), v(x)dx$$

c) $V = M_{22}$

$$W = \{u \in M_{22} : u^T = u\} \qquad < u, v > \equiv Tr(u^T v)$$

4. (5) Let $V = P_2[0,1]$ and define

$$< u, v >= u(0)v(0) + u(1)v(1)$$

Is this an inner product on V? State all axioms (if any) that fail.

5. (5) Recall the Fredholm alternative Theorem $N(A^T) = col(A)^\perp$ or,

Theorem 1 *Let $A \in \mathbb{R}^{n\times n}$. Then, $Ax = b$ has a solution $\Leftrightarrow < v, b >= 0$, $\forall v \in N(A^T)$.*

Use this theorem to determine for what $\alpha \in \mathbb{R}$ (if any) the following system has a solution:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= \alpha \\ 3x_1 + x_3 &= 1 + \alpha \\ x_1 - x_2 + x_3 &= 2 + 3\alpha \end{aligned}$$

6. (5) Here $V = P_2[-1,1]$ and $< u, v >= \int_{-1}^{1} u(x)v(x)dx$. The set

$$S = \{v_1, v_2, v_3\} = \left\{ \frac{1}{\sqrt{2}} \ , \ \frac{\sqrt{6}}{2}x \ , \ \frac{\sqrt{10}}{4}(3x^2 - 1) \right\}$$

is an orthonormal basis for V. These are the first three normalized Legendre Polynomials used in Physics. Find $(w)_S$ if $w = \sqrt{18}\left(x^2 + x + 1\right)$. Simplify radicals.

7 (5) For the indicated inner product spaces V, subspaces W and vector u, compute the projection $w = proj_W u$ and $w^\perp$ in the orthogonal decomposition $u = w + w^\perp$.

$$\begin{aligned} V &= P_3 \quad , \quad < u, v >\equiv \int_0^1 u(x)v(x)dx \quad , \\ u &= x^3 - x \quad , \quad W = span\{x^3, x^2 + x\} \end{aligned}$$