Math 333: Homework 5: Due: Tuesday, Dec. 1, 2015 in class

1. [5pt] A basis for $V = \mathbb{R}^3$ is

$$S = \{u_1, u_2, u_3\} = \{(2,1,0), (1,-1,1), (4,1,1)\}$$

Use the Gram-Schmidt process to derive an orthogonal basis $S_\perp$.

2. [5pt] Let $V = \mathbb{R}^3$ and

$$W = span\,\{u_1, u_2\} = span\,\{(1,1,1), (0,1,-1)\}$$

be a subspace defined by its orthogonal basis vectors u_k. Find a matrix $A \in \mathbb{R}^{3\times 3}$ such that

$$T(v) \equiv Av = proj_W v$$

An example of this procedure is given in the posted notes.

3. [5pt] We are given the data points

$$Z = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\} = \{(-1,2), (0,1), (1,3)\}$$

and want a least squares fit to the quadratic model

$$y = ax^2 + c$$

Thus, $\mathbf{x} = (a, c)$ must be the least squares solution of

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} x_1^2 & 1 \\ x_2^2 & 1 \\ x_3^2 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Find $\mathbf{x}$ and plot the data along with $y = ax^2 + c$.

4. [5pt] Let $\hat{x}$ be the least squares solution of $Ax = b$ and define the *residual* vector $r \equiv A\hat{x} - b$. Prove $\| r \|^2 = -< b, r >$. To do so, start noting

$$\| r \|^2 = (A\hat{x} - b)^T (A\hat{x} - b) \cdots$$

Write this out as a formal proof explaining each step.

5. [5pt] Let $T : X \to \mathbb{R}^3$ be a linear transformation with basis $S = \{u_1, u_2, u_3\}$. Given

$$T(u_1) = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \quad T(u_2) = \begin{pmatrix} -5 \\ 7 \\ -1 \end{pmatrix} \quad T(u_3) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Find a basis for the range $R(T)$. DO NOT put $T(u_k)$ in the rows of any matrix.

6. [25pt] In each the the four problems below a linear transformation $T : X \to Y$ is defined. For each, clearly define the kernel $ker(T)$ and the range $R(T)$ of each transformation. When T has an inverse, define its formula and its domain.

a) $T : P_5 \to \mathbb{R}$ (average)

$$T(u) = \int_{-1}^{1} u(x)dx$$

b) $T : P_2 \to P_2$

$$T(u) = u(x+1) - u(x-1)$$

c) $T : M_{22} \to \mathbb{R}$

$$T(u) = trace(u) \qquad , \qquad u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

d) $T : P_2 \to P_2$

$$T(u) = x\frac{du}{dx}$$

e) $T : P_2 \to P_2$

$$T(u) = u(2x+1)$$