Linear Systems

m equations for n unknowns $x_1, x_2, \dots x_n$

$$\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m \end{array}}$$

Generally there are three possibilities

(1) unique solution

(2) infinitely many solutions

(3) no solution

EXAMPLE (unique solution)

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7/3 \\ 4/3 \end{pmatrix}$$

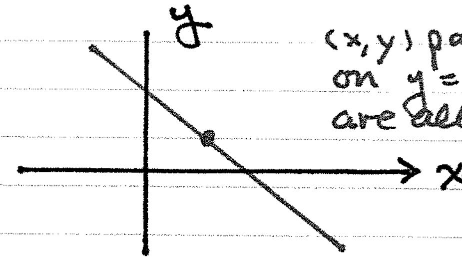
EXAMPLE (no solution)

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 0 \end{aligned} \qquad \text{(divide by 2)}$$

Hence $x+y=1$ and $x+y=0$ is impossible

EXAMPLE (infinite solns)

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \end{aligned}$$

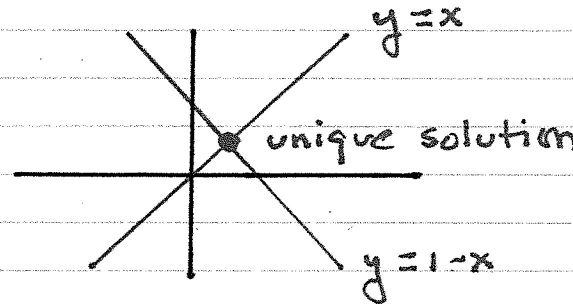


(x, y) pairs on $y = 1 - x$ are all solns.

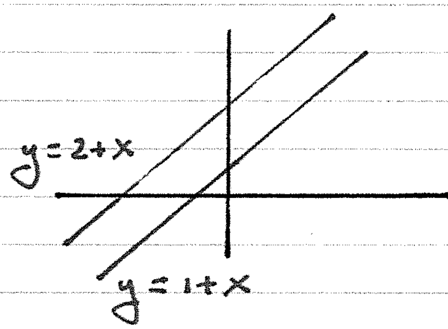
Geometrical considerations for $x \in \mathbb{R}^2$

For $x = (x_1, x_2)$ each equation describes a line in $\mathbb{R}^2$

$$\boxed{\begin{aligned} x + y &= 1 \\ -x + y &= 0 \end{aligned}}$$



$$\boxed{\begin{aligned} -x + y &= 1 \\ -x + y &= 2 \end{aligned}}$$



no solution

Digress: Solution space using set notation

For the first example above

$$S \equiv \{ x \; : \; x_1 + x_2 = 1 \, , \, -x_1 + x_2 = 0 \}$$

is the set of x that solve both equations. One can solve the system to show

$$S = \{ (\tfrac{1}{2}, \tfrac{1}{2}) \} \qquad \text{sole element}$$

For the second example

$$S = \phi \qquad \text{"empty set } \phi \text{"}$$

Geometrical considerations $x \in \mathbb{R}^3$

$$a_1 x + b_1 y + c_1 z = d_1$$

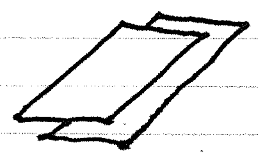
$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3$$

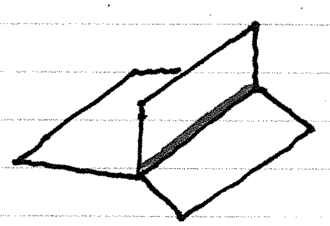
Each equation describes a plane. The system has a unique solution only if the planes intersect at a point. Normal vectors for plane are

$$\vec{N}_K = (a_K, b_K, c_K)$$

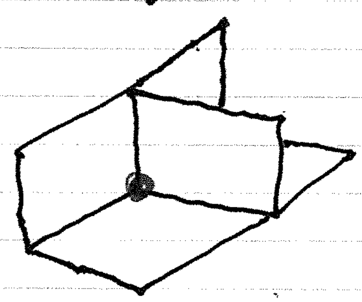
A few examples



2 parallel planes
$\Rightarrow$ no solution



3 planes intersecting on a line
$\Rightarrow$ infinetly many solns



3 planes intersect at a point
$\Rightarrow$ unique soln.

Matrix Notation

Let $x, y \in \mathbb{R}^n$ be vectors and $A, B \in \mathbb{R}^{m\times n}$ be matrices.

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad x^T = (x_1, \ldots x_n)$$

column vector   row vector

For a matrix with m rows and n vectors

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

Matrices are composed of row/column vectors

$$A = \begin{bmatrix} \text{---} & r_1 & \text{---} \\ \text{---} & r_2 & \text{---} \\ & \vdots & \\ \text{---} & r_m & \text{---} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix}$$

Summation Notations

Dot product $\qquad x^T y = \sum_{i=1}^{n} x_i y_i$

Matrix product $\qquad$ Let $[Ax]_i$ = i^{th} element of Ax

$$[Ax]_i = r_i^T x = \sum_{j=1}^{n} a_{ij} x_j$$

Gaussian Elimination

A system of m equations can be represented by a matrix equation

$$Ax = b \qquad\qquad A \in \mathbb{R}^{m \times n}$$

To find solutions (if they exist) one can use Gaussian Elimination to row reduce the augment matrix

$$[A \mid b] \underset{\text{row reduce}}{\sim} \underset{\text{upper (triangular) echelon.}}{[U \mid c]}$$

Row operations can include scaling rows and permuting rows.

Since U is upper echelon the system

$$Ux = c$$

can easily be back solved to find x.

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & & * & * & * \\ 0 & 0 & 0 & & * \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & \boxed{0} & * & \boxed{0} & * \\ 0 & 1 & * & \boxed{0} & * \\ 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Row Echelon $\qquad\qquad$ Reduced Row Echelon

EXAMPE (UNIQUE SOLN)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 3 & 1 & 1 \end{bmatrix} \qquad b = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$$

One can verify $\bigcirc$ = pivot

$$[A \mid b] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 2 \\ 0 & 0 & -3 & 3 \end{array}\right] = [U \mid c]$$

Backsolve to get $x = (2, 0, -1)^T$

EXAMPLE (NONUNIQUE SOLN)

An augmented matrix already in upper echelon form

$$\begin{array}{cc} & \begin{array}{cccc} & & \text{free variable} \downarrow & \end{array} \\ [A \mid b] = & \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$\bigcirc$ = pivot

Solve these. For any z a soln is

$$x = -1 - 3z$$
$$y = 2 + 4z$$

z "free

Let $z = t$ then "solution" is a line in $\mathbb{R}^3$.

$$\vec{r}(t) = (-1-3t, 2+4t, t)$$

nonunique. Any point on line's a solution

EXAMPLE (NO SOLUTION) with permutations

$$[A \mid b] = \begin{bmatrix} 0 & 0 & -2 & 1 \\ 2 & 4 & -10 & 5 \\ 2 & 4 & -5 & 5 \end{bmatrix} \quad \text{zero pivot}$$

$$\sim \begin{bmatrix} 2 & 4 & -10 & 5 \\ 0 & 0 & -2 & 5 \\ 2 & 4 & -5 & 5 \end{bmatrix}$$

$$\Rightarrow \quad \sim \begin{bmatrix} 2 & 4 & -10 & 5 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \quad \text{row echelon}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{reduced row echelon}$$

Note that the equation $\Rightarrow$ indicated above states

$$0x_1 + 0x_2 + 0x_3 = \frac{5}{2}$$

which is clearly not possible for any $x = (x_1, x_2, x_3)$

Numerical Issues with Gaussian Elimination

Let $\varepsilon \ll 1$ be small and consider

$$(1) \qquad \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \bigcirc \text{ pivot}$$

Augmented row reduction

$$\begin{bmatrix} \varepsilon & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} \varepsilon & 1 & 1 \\ 0 & (1-\varepsilon^{-1}) & (2-\varepsilon^{-1}) \end{bmatrix}$$

Backsolve with finite arithmetic (ε^{-1} is huge)

$$x_2 = \frac{2-\varepsilon^{-1}}{1-\varepsilon^{-1}} \simeq 1$$

$$x_1 = \varepsilon^{-1}(1-x_2) \simeq 0$$

Thus it would appear $\bar{x} = (0,1)^T$ is a solution (or close too) of $Ax = b$ eqn (1) above. However

$$A\bar{x} = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 2 \end{pmatrix} = b$$

Turns out a better approximation is $\bar{x} = (1,1)^T$ using the alternative pivot below

$$\begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

yields

$$x_1 = 2 - x_2 \qquad x_2 = \frac{1-2\varepsilon}{1-\varepsilon}$$

Gauss elimination with partial pivoting

Chose pivot having largest magnitude.

By way of example

$$\begin{bmatrix} 2 & 3 & 101 \\ 1 & 1 & 25 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 8 \end{bmatrix}$$

Row reduce augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 3 & 101 & 8 \\ 1 & 1 & 25 & 4 \\ 4 & 2 & 2 & 8 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 0 & 2 & 100 & 4 \\ 0 & \frac{1}{2} & \frac{49}{2} & 2 \\ 4 & 2 & 2 & 8 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 0 & 2 & 100 & 4 \\ 0 & 0 & -\frac{1}{2} & 1 \\ 4 & 0 & -98 & 4 \end{array}\right]$$

From which $x = (-48, 102, -2)$.

<u>Remarks</u>: (1) Zero pivots are avoided.

(2) Result is a permutation of an upper echelon matrix

Matrices and Matrix Multiplication

Definition: A matrix is an array of elements

Here elements can be from the following sets

$\mathbb{N}$	natural numbers	$(0 \in \mathbb{N})$
$\mathbb{Z}$	integers	
$\mathbb{Q}$	rational numbers	
$\mathbb{R}$	real numbers	
$\mathbb{C}$	complex numbers	
P_n	set of all n^{th} degree polynomials	

EXAMPLE

$$A = \begin{bmatrix} \pi & 2 \\ \sqrt{2} & 7 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

$$A = \begin{bmatrix} i & 1 \\ 0 & 2+i \end{bmatrix} \in \mathbb{C}^{2\times 2}$$

EXAMPLE Let $S \in \{1, 2, 3\}$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \qquad A, B \in S^{2\times 2}$$

but $AB \notin S^{2\times 2}$ since product has elements not in S.

EXAMPLE

$$A = \begin{bmatrix} x & x^2+1 \\ x+3 & 1-x^2 \end{bmatrix} \in P_2^{2\times 2}$$

Matrix Multiplication

Let $A \in \mathbb{R}^{m \times p}$ and $B \in \mathbb{R}^{p \times n}$ then $AB \in \mathbb{R}^{m \times n}$.

$$A = \begin{bmatrix} \text{---} & a_1 & \text{---} \\ \text{---} & a_2 & \text{---} \\ & \vdots & \\ \text{---} & a_m & \text{---} \end{bmatrix} \qquad B = \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{bmatrix}.$$

row vectors of A column vectors of B

Then the product

$$C = AB \qquad C \equiv [c_{ij}]$$

where c_{ij} are elements of C are computed:

$$c_{ij} = a_i \cdot b_j \qquad \text{dot product}$$

EXAMPLE $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 4 & 4 \\ 2 & 4 \end{bmatrix} \qquad BA \text{ undefined!!}$$

EXAMPLE Non commuting square matrices

$$AB = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{B} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad AB \neq BA$$

$$BA = \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Matrix multiplication : linear combination of columns

Let $A \in \mathbb{R}^{m \times p}$ and $x \in \mathbb{R}^{p \times 1}$ (column vector)

$$A = [a_{ij}]$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$$

Note that Ax is defined and $Ax \in \mathbb{R}^m$

View A as a set of column vectors

$$A = \begin{bmatrix} | & | & & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_p \\ | & | & & | \end{bmatrix}$$

One can easily verify

$$\boxed{Ax = x_1 \vec{c}_1 + x_2 \vec{c}_2 + \cdots x_p \vec{c}_p}$$

which explicitly shows Ax is a <u>linear combination of the columns of A</u>.

<u>EXAMPLE</u>

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

AB as a column expansion

Let $A \in \mathbb{R}^{m \times p}$, $B \in \mathbb{R}^{p \times n}$ and b_k be the column vectors of B

column vectors of AB

$$AB = A[b_1\ b_2 \ldots b_n] = [Ab_1\ Ab_2 \ldots Ab_n]$$

EXAMPLE

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 10 & 6 \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ Ab_1 & Ab_2 \end{array}$$

AB as a row expansion

As above exect a_k are <u>row</u> vectors of A

$$AB = \begin{bmatrix} - a_1 - \\ - a_2 - \\ - a_m - \end{bmatrix} B = \begin{bmatrix} a_1 B \\ a_2 B \\ a_m B \end{bmatrix}$$ ← row vectors ← of product ← AB

EXAMPLE

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} (1,3)\,B \\ (2,4)\,B \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 10 & 6 \end{bmatrix}$$

where, for instance,

$$a_1 B = [1\ \ 3]\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = [7\ \ 4]$$

Matrices and Summation Notation Conventions:

In the following $A,\ B \in \mathbb{R}^{n\times n}$ are square matrices, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y}, \mathbf{b} \in \mathbb{R}^n$. In component (row vector) form

$$\begin{aligned} x &= (x_1, x_2, \ldots x_n) \\ y &= (y_1, y_2, \ldots y_n) \end{aligned}$$

The dot product of these vectors is:

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots x_n y_n = \sum_{i=1}^{n} x_i y_i$$

We simplify this cumbersome notation using the Einstein summation convention that expressions containing repeated indices have an implied sum associated with them. For instance, the dot product is simply

$$x \cdot y = x_i y_i$$

Since the index i is repeated

$$x_i y_i = x_1 y_1 + x_2 y_2 + \cdots x_n y_n$$

Likewise, for matrices where $A = [a_{ik}]$ we have

$$a_{ik}\, x_k = a_{i1} x_1 + a_{i2} x_2 + \cdots a_{in} x_n$$

since k is a repeated index. Note this is just the i^{th} component of Ax. We summarize with a few examples of matrix expressions and their index notation counterparts.

x	x_i
A	a_{ij}
A^T	a_{ji}
$x \cdot y$	$x_i y_i$
$Ax = b$	$a_{ik}\, x_k = b_i$
$A^T x$	$a_{ki}\, x_k$
AB	$a_{ij} b_{jk}$
$trace(A)$	a_{ii}

SPECIAL MATRICES (Square)

Upper Triangular

$$T = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

Diagonal matrices

$$D = \mathrm{diag}(a_1, \ldots a_n) = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix}$$

Identity matrices

$$I_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

When the dimensions are clear $I_n = I$
Most important property

$$AI = IA = A \qquad A \in \mathbb{R}^{n\times n}$$

Remarks

(1) Products of triangular and diagonal matrices are triangular and diag, respectively, i.e.

$$T_1 T_2 = T_3$$

$$D_1 D_2 = D_3$$

Transpose of $A \in \mathbb{R}^{m \times n}$

If $A = [a_{ij}]$ then $A^T = [a_{ji}]$.

This means the rows of A become the columns of A^T

EXAMPLE

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 3 & 5 \end{bmatrix}$$

Some algebraic properties (without proof)

$$(A+B)^T = A^T + B^T$$

$$(cA)^T = cA^T \qquad c \in \mathbb{R}$$

$$(AB)^T = B^T A^T$$

EXAMPLE: Verify last one for certain A, B

$$\Bigg(\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{B}\Bigg)^T = \underbrace{\begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix}}_{AB}^T = \underbrace{\begin{bmatrix} 3 & 5 \\ -1 & -1 \end{bmatrix}}_{(AB)^T} = B^T A^T$$

Remark Regardless of dimensions AA^T is square

$$AA^T \in \mathbb{R}^{m \times m} \qquad A^T A \in \mathbb{R}^{n \times n}$$