Inner Products

Defn Let V be a vector space and $\langle u, v \rangle$ be a function on V with the following properties

(B1) $\langle u, v \rangle = \langle v, u \rangle \qquad \forall u, v \in V$

(B2) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

(B3) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle \qquad \forall \alpha \in \mathbb{R}$

(B4) $\langle u, u \rangle \geq 0$

(B5) $\langle u, u \rangle = 0 \iff u = 0$

Then $\langle u, v \rangle$ is an innerproduct on V

Defn Any V with an inner product is a (real) inner product space

Remark Sometimes $\langle u, v \rangle$ is referred to as a bilnear map.

$$T : V \times V \to \mathbb{R}$$

and is a linear map if each argument fixed

$$T_v(u) \equiv \langle u, v \rangle \qquad v\text{-fixed}$$

$$T_u(v) \equiv \langle u, v \rangle \qquad u\text{-fixed}$$

EXAMPLE Euclidean inner product $V = \mathbb{R}^n$

$$\langle u, v\rangle = u_1 v_1 + \cdots u_n v_n$$

EXAMPLE Weighted Euclidean inner prod. $V = \mathbb{R}^n$

$$\langle u, v\rangle = \alpha_1 u_1 v_1 + \cdots \alpha_n u_n v_n$$

where $\alpha_1, \ldots, \alpha_n$ are fixed (strictly) positive const.

EXAMPLE Polynomial spaces $V = P_n$

$$\langle u, v\rangle = a_0 b_0 + a_1 b_1 + \cdots a_n b_n$$

where $u(x) = a_0 + a_1 x + \cdots a_n x^n$ and $v(x) = b_0 + b_1 x + \cdots b_n x^n$

EXAMPLE Let $A \in \mathbb{R}^{n\times n}$ be invertible and $V = \mathbb{R}^n$

$$\langle u, v\rangle \equiv (Au)^T (Av)$$

Since $\langle u, v\rangle \in \mathbb{R}$ it equals its own transpose:

$$\begin{aligned}\langle u, v\rangle &= \left((Au)^T(Av)\right)^T \\ &= (Av)^T\left((Au)^T\right)^T \\ &= (Av)^T (Au) \\ &= \langle v, u\rangle\end{aligned}$$

this proves axiom (B1). We don't show (B2)-(B3) but $\langle u, u\rangle = (Au)^T(Au) \geq 0$. Moreover

$$\langle u, u\rangle = 0 \iff \|Au\|^2 = 0$$

$\Rightarrow u = 0$ since A is invertible. Thus proving (B4)-(B5)

EXAMPLE $V = M_{22}$

$$\langle u, v\rangle = Tr(u^T v) = Tr(v^T u) = \langle v, u\rangle$$

where $Tr(A)$ is the trace of A. This is defined to be the sum of the diagonals. Equality above arises from the identity

$$Tr(A) = Tr(A^T) \qquad A = u^T v$$

Property (B3)

$$\begin{aligned}\langle \alpha u, v\rangle &= Tr\left((\alpha u)^T v\right)\\ &= Tr(\alpha u^T v)\\ &= \alpha\, Tr(u^T v)\\ &= \alpha \langle u, v\rangle\end{aligned}$$

The latter identities follow from

(B4) $\langle u, u\rangle = Tr(u^T u)$

$$= u_{11}^2 + u_{12}^2 + u_{21}^2 + u_{22}^2 \geq 0$$

and

$$\langle u, u\rangle = 0 \iff u_{ij} = 0 \text{ for all } i, j$$

EXAMPLE **Function Spaces** $V = C[a,b]$

$$\langle u, v \rangle \equiv \int_a^b u(x)v(x)\,dx$$

Clearly $\langle u,v\rangle = \langle v,u\rangle \Rightarrow$ (B1) true.
Also, (B2)-(B3) are trivial. Then

$$\langle u, u\rangle = \int_a^b u(x)^2\,dx \geq 0$$

and equality only if (continuous) $u(x) \equiv 0$ on $[a,b]$.

EXAMPLE **Failure** $V = \mathbb{R}^2$

$$\langle u, v\rangle = (Au)\cdot(Av) \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Most axioms are true but (B5) not satisfied

$$\langle u, u\rangle = (Au)\cdot(Au)$$
$$= u_1^2 \geq 0$$

However $\langle u,u\rangle = 0$ for all $u = (0, u_2)$, not just $\vec{0}$.

Inner Product Norms

If V is an inner product then the inner product can be used to define a norm. This is referred to as the inner product induced norm given by:

$$\boxed{\|u\| \equiv \langle u, u\rangle^{1/2}}$$

It obeys all the norm axioms (without proof)

(N1) $\quad \|u\| \geq 0$

(N2) $\quad \|u\| = 0 \iff u = 0$

(N3) $\quad \|ku\| = |k|\,\|u\| \qquad \forall k \in \mathbb{R}$

(N4) $\quad \|u+v\| \leq \|u\| + \|v\| \qquad$ (triangle. ineq.)

The last can be used to a metric (distance)

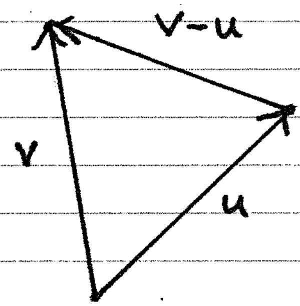
$$\boxed{d(u,v) \equiv \|u - v\|}$$

Appendix (Pf of N4)

$$\begin{aligned}
\|u+v\|^2 &= \|u\|^2 + 2\langle u,v\rangle + \|v\|^2 \\
&\leq \|u\|^2 + 2|\langle u,v\rangle| + \|v\|^2 \\
&\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \\
&\leq (\|u\| + \|v\|)^2
\end{aligned}$$

Cauchy Schwartz

Ex $u = (1,2,3)$ $v = (4,3,4)$ $V = \mathbb{R}^3$



$v - u = (3,1,1)$

$\|v-u\| = \sqrt{3^2+1^2+1^2} = \sqrt{11}$

$\langle u, v \rangle = 4 + 6 + 12 = 22$

Ex $u = (1,2,3)$ $v = (4,3,4)$ on $V = \mathbb{R}^3$
but with the weighted inner product

$$\langle u, v \rangle \equiv 2u_1v_1 + \tfrac{1}{2}u_2v_2 + u_3v_3$$

Then

$$\langle u, v \rangle = 2(1)(4) + \tfrac{1}{2}(2)(3) + (3)(4) = 23$$

Then we can calculate the weighted distance

$$\|v-u\|^2 = \langle v-u, v-u \rangle \qquad v-u = (3,1,1)$$

$$= 2(3)^2 + \tfrac{1}{2}(1)^2 + (1)^2$$

$$= 19\tfrac{1}{2}$$

Hence

$$\|v-u\| = \frac{39}{2}$$

EXAMPLE $\quad V = P_2 \qquad \langle u, v \rangle \equiv \int_0^1 u(x)v(x)\,dx$

Some sample calculations with

$$u(x) = 5x^2 + x \qquad v(x) = x^2 + 3x$$

$$\|u\|^2 = \int_0^1 (5x^2+x)^2\,dx = \frac{47}{6}$$

$$\|u\| = \sqrt{\frac{47}{6}} \qquad \text{"length of } u\text{"}$$

and distance: $\quad w = u - v = 4x^2 - 2x$

$$\langle u-v, u-v \rangle = \int_0^1 (4x^2-2x)^2\,dx = \frac{8}{15}$$

$$\|u-v\|^2 = \frac{8}{15}$$

hence

$$d(u,v) = \|u-v\| = \sqrt{\frac{8}{15}} \qquad \text{"distance between } u, v\text{"}$$

Or a regular inner product

$$\langle u, v \rangle = \int_0^1 (5x^2+x)(x^2+3x)\,dx = \cdots\cdots = 6$$

EXAMPLE $V = M_{22}$ $\langle u, v \rangle \equiv \underset{\uparrow \text{ Trace}}{\mathrm{Tr}}(u^T v)$

$$u = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad v = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

Here

$$u^T v = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & * \\ * & 4 \end{bmatrix} \Rightarrow \langle u, v \rangle = 4$$

Now an example of a norm/distance

$$d(u,v) = \|u - v\|$$

First

$$u - v = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$$

$$(u-v)^T(u-v) = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\|u - v\|^2 = \langle u-v, u-v \rangle = 6$$

to conclude

$$d(u,v) = \sqrt{6}$$

Orthogonality

Defn Let V be an inner product space.
Then $u, v \in V$ are orthogonal $\Leftrightarrow \langle u, v \rangle = 0$.

Ex $V = \mathbb{R}^3$ $\quad \langle u, v \rangle \equiv u_1 v_1 + u_2 v_2 + u_3 v_3$

$$u = (0, 2, 1)$$

$$v = (1, -1, 2)$$

$$\langle u, v \rangle = 0 \quad \Rightarrow \quad u \perp v$$

Ex $V = \mathbb{R}^3$ $\quad \langle u, v \rangle \equiv u_1 v_1 + 2 u_2 v_2 + u_3 v_3$

Using same u, v above

$$\langle u, v \rangle = 0 - 4 + 2 = -2 \neq 0$$

Hence orthogonality is tied to which inner product you are using

Ex $V = C[0, \pi]$ $\quad \langle u, v \rangle = \int_0^\pi u(x) v(x)\, dx$

$$u = \sin x \qquad v = \cos x$$

hence

$$\| u \|^2 = \int_0^\pi \sin^2 x \, dx = \frac{\pi}{2} \quad \Rightarrow \quad \| u \| = \sqrt{\frac{\pi}{2}}$$

$$\langle u, v \rangle = \int_0^\pi \sin x \cos x \, dx = 0 \quad \Rightarrow \quad u \perp v$$

EXAMPLE $V = M_{22}$ $\langle u, v \rangle = \mathrm{Tr}(u^T v)$

For what α are u, v orthogonal if

$$u = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \qquad v = \begin{bmatrix} \alpha & 1 \\ 1 & 3 \end{bmatrix} \qquad \alpha \in \mathbb{R}$$

First

$$u^T v = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \alpha+1 & 4 \\ 2 & 6 \end{bmatrix}$$

$$\langle u, v \rangle = \mathrm{Tr}(u^T v) = \alpha + 7 = 0 \iff \alpha = -7$$

EXAMPLE For what $\alpha \in \mathbb{R}$ are $u(x) = \alpha - x$ and $v(x) = x$ orthogonal in $V = C[0,1]$ if

$$\langle u, v \rangle = \int_0^1 u(x) v(x)\, dx$$

$$\langle u, v \rangle = \int_0^1 x(\alpha - x)\, dx = \frac{1}{2}\alpha - \frac{1}{3} = 0$$

only if

$$\alpha = \frac{2}{3}$$

Orthogonal Complements

Defn: Let W be a subspace of V.
The orthogonal complement
$W^\perp$ of W is defined:

$$W^\perp \equiv \{ v \in V : \langle u, v \rangle = 0, \ \forall u \in W \}$$

In words, any $w \in W^\perp$ is orthogonal to every element of W.

Theorem Let $\dim V = n < \infty$ and W a subspace of V.

(a) $W^\perp$ is a subspace of V

(b) $W \cap W^\perp = \{0\}$

(c) $(W^\perp)^\perp = W$

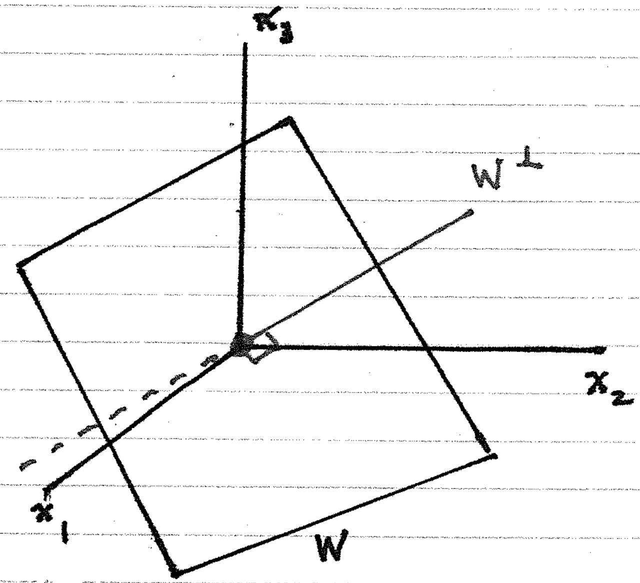
Proof (a): Let $u, v \in W^\perp$, $k \in \mathbb{R}$

$$\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle = 0 \quad \text{closure } +$$
$$\langle ku, w \rangle = k \langle u, w \rangle = 0 \quad \text{closure } .$$

Proof (c) Let $u \in W \cap W^\perp$ then

$$\langle u, u \rangle = \| u \|^2 = 0 \quad \Rightarrow u = 0.$$

Geometry of $W^\perp$ in $V = \mathbb{R}^3$ (Euclidean inner prod)



$W = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$

is a plane thru origin. For this plane P

$$\vec{N} = (1, 1, 1)$$

is orthogonal to P.

$W^\perp = \operatorname{span}\{\vec{N}\}$

EXAMPLE Suppose $v_1 = (1, 1, 1)$, $v_2 = (-1, 2, 3)$ Then

$$W \equiv \operatorname{span}\{v_1, v_2\}$$

is a plane thru the origin. Next

$$\vec{N} \equiv v_1 \times v_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ -1 & 2 & 3 \end{vmatrix} = (1, -4, 3)$$

Orthogonal complement

$$W^\perp = \operatorname{span}\{\vec{N}\}$$

In particular, note

$$\langle \vec{N}, v_k \rangle = 0 \qquad k = 1, 2$$

EXAMPLE $V = P_2[0,1]$ inner product space where

$$\langle u, v \rangle \equiv \int_0^1 u(x)v(x)\,dx$$

Next define the subspace

$$W \equiv \mathrm{span}\{v_1, v_2\} = \mathrm{span}\{1, x\}$$

Find a basis for $W^\perp$.

To do so we note any $w^\perp \in W^\perp$ must be orthogonal to both v_1 and v_2. Let

$$w^\perp(x) = a + bx + cx^2$$

$$(w^\perp)_E = (a, b, c) \qquad \text{standard coordinate}$$

Some calculations reveal:

$$\langle v_1, w^\perp \rangle = a + \tfrac{1}{2}b + \tfrac{1}{3}c = 0$$

$$\langle v_2, w^\perp \rangle = \tfrac{1}{2}a + \tfrac{1}{3}b + \tfrac{1}{4}c = 0$$

Thus $(w^\perp)_E \in N(A)$ where

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 1 \end{bmatrix} \qquad \dim N(A) = 1$$

Since $N(A) = \mathrm{span}\{(1, -6, 6)\}$,

$$W^\perp = \mathrm{span}\{w^\perp\} \qquad w^\perp(x) = 6x^2 - 6x + 1$$

EXAMPLE $\quad V = M_{22} \qquad \langle u, v\rangle \equiv Tr(u^T v)$

Seek $W^\perp$ given $W = span\{v_1, v_2\}$ where

$$v_1 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Let $w^\perp \in W^\perp$ be given by

$$w^\perp = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$w^\perp$ must be orthogonal to v_1 and v_2.
After some calculations

(1) $\quad \langle v_1, w^\perp\rangle = a + 2b + c + d \quad = 0$

(2) $\quad \langle v_2, w^\perp\rangle = a + b - c + d \quad = 0$

Thus $(a, b, c, d) \in N(A)$ where the matrix A

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$\uparrow \ \uparrow$ free

A basis for N(A) is given by

$$x_1 = (0, -2, +1, 0) \qquad x_2 = (-1, 0, 0, 1)$$

Thus

$$W^\perp = span\{w_1, w_2\} \qquad w_1 = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

EXAMPLE $V = \mathbb{R}^4$ Euclidean inner product

$$W \equiv \text{span}\{v_1, v_2\} = \text{span}\{(1,1,0,1)^T, (2,1,1,1)^T\}$$

To find a basis for $W^\perp$ we let $\omega^\perp \in W^\perp$

$$\omega^\perp = (a, b, c, d)^T$$

The requirements $\langle v_k, \omega^\perp \rangle = 0$ for $K = 1, 2 \Rightarrow$

$$\begin{aligned} a + b \quad\quad + d &= 0 \\ 2a + b + c + d &= 0 \end{aligned}$$

So $\omega^\perp \in N(A)$ where

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

A basis for $N(A)$

$$\omega_1^\perp = (0, -1, 0, 1)^T \qquad \omega_2^\perp = (1, -1, 0, 0)^T$$

So that

$$W^\perp = \text{span}\{\omega_1^\perp, \omega_2^\perp\}$$

$$\dim W^\perp = 2$$

Also note

$$V = \text{span}\{v_1, v_2, \omega_1^\perp, \omega_2^\perp\}$$

can be verified.

Orthogonality of Matrix Spaces

Theorem Let $A \in \mathbb{R}^{m \times n}$ and $\langle u, v \rangle$ be the Euclidean inner product. Then

(a) $N(A) = \text{row}(A)^{\perp}$

(b) $N(A^T) = \text{col}(A)^{\perp}$

Proof of (b)

Let $v \in N(A^T)$ and $b \in \text{col}(A)$. Then $\exists x$ s.t.

take transp.

$$Ax = b$$
$$v^T A x = v^T b$$
$$x^T \underline{A^T v} = \langle v, b \rangle = 0$$

This shows $v \in N(A^T) \Rightarrow v \in \text{col}(A)^{\perp} \Rightarrow$ $\underline{N(A^T) \subset \text{col}(A)^{\perp}}$

Now let $v \in \text{col}(A)^{\perp}$. Then

transp.

$$v^T A x = 0 \qquad \forall x \in \mathbb{R}^n$$
$$x^T (A^T v) = 0 \qquad \forall x \in \mathbb{R}^n$$
$$A^T v = 0$$

Thus $\text{col}(A^T) \subset N(A^T)$ and we conclude

$$N(A^T) = \text{col}(A)^{\perp} \qquad \square$$

Fredholm Alternative on $\mathbb{R}^n$

Theorem Let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$.

$$\underset{\text{has a solution}}{Ax = b} \iff \langle v, b\rangle = 0 \quad \forall v \in N(A^T)$$

Proof $\Rightarrow$

Suppose $Ax=b$ has a soln. Then $b \in col(A)$. But previous Thm $\Rightarrow$ $b \in N(A^T)^\perp$. Hence $\langle v,b\rangle = 0 \quad \forall v \in N(A^T)$

Proof $\Leftarrow$

If $\langle v,b\rangle = 0 \quad \forall v \in N(A^T)$ then $b \in N(A^T)^\perp$

$$b \in N(A^T)^\perp = (col(A)^\perp)^\perp = col(A)$$

so that $Ax=b$ has a solution.

EXAMPLE For what α (if any) does

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} x = \begin{bmatrix} \alpha \\ \alpha+1 \end{bmatrix} = b$$

have a solution. Here we can show

$$N(A^T) = \operatorname{span}\{v_1\} \qquad v_1 = (2,-1)^T$$

Thus

$$\langle v_1, b\rangle = \alpha - 1 = 0$$

only if $\alpha = 1$. $N(A) \neq \{0\} \Rightarrow$ soln not unique.

Eigenspaces of Symmetric Matrices $\quad A = A^T \in \mathbb{R}^{n \times n}$

One fact about symmetric matrices is that eigenvectors corresponding to different eigenvalues are orthogonal. This is true using the standard Euclidean inner product.

We prove this here. Let λ_1, λ_2 be e-values.

$$A v_1 = \lambda_1 v_1$$

$$A v_2 = \lambda_2 v_2$$

Now consider

(1) $$\langle v_2, A v_1 \rangle = \lambda_1 \langle v_2, v_1 \rangle$$

(2) $$\langle v_1, A v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$$

The terms on the left of (1)-(2) are equal:

$$\langle v_2, A v_1 \rangle^T = (v_2^T A v_1)^T = v_1^T A v_2 = \langle v_1, A v_2 \rangle$$

Thus, subtracting

$$(\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle = 0$$

Since $\lambda_1 \neq \lambda_2$

(3) $$\langle v_1, v_2 \rangle = 0$$

Eigenspace Definition

The eigenspace $E_\lambda(A)$ of A with eigenvalue λ is

$$E_\lambda(A) = \{x : Ax = \lambda x\} = N(A - \lambda I)$$

Symmetric case:

If $\lambda_1 \neq \lambda_2$ and $v_1 \in E_{\lambda_1}(A)$ then $\langle v_1, v_2 \rangle = 0$ for all $v_2 \in E_{\lambda_2}(A)$ by our previous discussion:

$$E_{\lambda_1}(A) \subset E_{\lambda_2}(A)^\perp$$

EXAMPLE

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Can show $P(\lambda) = (\lambda+3)(\lambda-3)^2$ charact. poly.
In this case calculations show $\lambda_1 = 3$, $\lambda_2 = -3$

$$W \equiv E_{\lambda_1}(A) = \text{span}\{v_1, v_2\} \qquad v_1 = (-1, 1, 0) \quad v_2 = (1, 0, 1)$$

$$E_{\lambda_2}(A) = \text{span}\{v_3\} \qquad v_3 = (-1, -1, 1)$$

Since $\mathbb{R}^3 = \text{span}\{v_1, v_2, v_3\}$

$$W^\perp = E_{\lambda_2}(A)$$