Orthogonal and Orthonormal Bases

A set $S = \{v_1, v_2 \ldots v_n\}$ is an orthogonal basis for V if

(i) S is a basis for V

(ii) $\langle v_i, v_j \rangle = 0 \qquad i \neq j$

The latter says that v_k are mutually orthogonal. Generally speaking $\|v_k\| \neq 1$ but if

(iii) $\langle v_i, v_i \rangle = 1 \qquad \forall i = 1, 2 \ldots n$

the set is said to be an orthonormal basis.

EXAMPLE $\quad V = \mathbb{R}^3$ orthonormal basis

$$S = \{(0,1,0), (\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}), (\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}})\} = \{v_1, v_2, v_3\}$$

mutually orthogonal under Euclidean inner product

EXAMPLE $\quad V = P_1$ orthonormal basis

$$S = \{1, \sqrt{12}(\tfrac{1}{2} - x)\} = \{v_1, v_2\}$$

using $\langle u, v \rangle = \int_0^1 u(x) v(x)\, dx$.

$$\langle v_1, v_1 \rangle = \int_0^1 1^2\, dx = 1$$

$$\langle v_1, v_2 \rangle = \int_0^1 \sqrt{12}(\tfrac{1}{2} - x)\, dx = \tfrac{1}{24}(\tfrac{1}{2} - x)^2 \Big|_1^0 = 0$$

and can verify $\langle v_2, v_2 \rangle = 1$.

EXAMPLE $V = M_{22}$ $\quad\quad\quad \langle u, v \rangle = Tr(u^T v)$

Consider the subset of symmetric matrices

$$W = \{ w \in V : w^T = w \}$$

The following is an orthogonal basis for W

$$S = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

It is not an orthonormal basis, since,

$$\langle v_2, v_2 \rangle = 2$$

Expansions and Orthogonal Coordinates.

Throughout, $S = \{v_1, \dots v_n\}$ will be an orthogonal basis for V. Then for any $v \in V$ there are constants $\alpha_1 \dots \alpha_n$ such that

$$(1) \qquad v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n$$

Thus, the coordinate of v relative to S is

$$(2) \qquad (v)_S = (\alpha_1, \dots \alpha_n)$$

Normally finding α_k involves solving an n by n system. But we can exploit the orthogonality of v_k to greatly reduce the amount of work to find $(v)_S$.

Take the inner product of v with v_k

$$\langle v, v_k \rangle = \alpha_1 \underbrace{\langle v_1, v_k \rangle}_{=0} + \cdots \alpha_k \underbrace{\langle v_k, v_k \rangle}_{\neq 0} + \cdots \alpha_n \underbrace{\langle v_n, v_k \rangle}_{=0}$$

Only the k^{th} term is nonzero. Thus

$$\boxed{\alpha_k = \frac{\langle v, v_k \rangle}{\langle v_k, v_k \rangle} = \frac{\langle v, v_k \rangle}{\|v_k\|^2}}$$

and

$$\boxed{v = \frac{\langle v, v_1 \rangle}{\|v_1\|^2} v_1 + \cdots \frac{\langle v, v_n \rangle}{\|v_n\|^2} v_n}$$

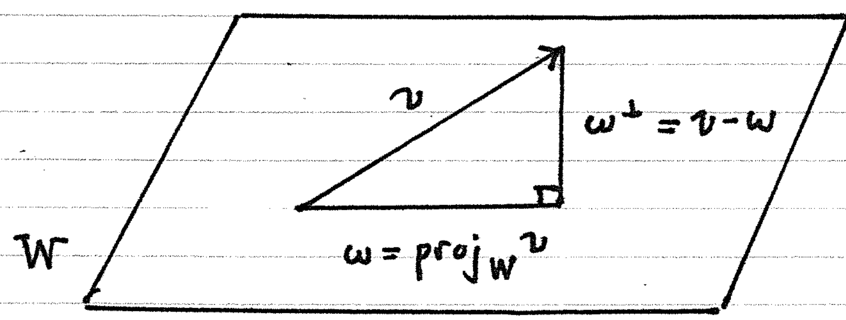
Projections

Let W be a subspace of V with orthonormal basis

$$S = \{v_1, v_2, \ldots v_n\}$$

so that $W = \text{span}(S) \subset V$. Then we define the projection of $v \in V$ onto W by

$$(1) \quad \omega = \text{proj}_W v = \langle v, v_1 \rangle v_1 + \langle v, v_2 \rangle v_2 + \cdots \langle v, v_n \rangle v_n$$



In what sense is this picture true given (1)? One would expect $v - \omega \in W^\perp$. We can verify this to be true:

$$\langle \omega - v, v_k \rangle = \sum_{i=1}^{n} \langle v, v_i \rangle \overbrace{\cancel{\langle v_i, v_k \rangle}}^{0 \text{ if } i \neq k} - \langle v, v_k \rangle$$

$$= \langle v, v_k \rangle \cdot 1 \quad - \langle v, v_k \rangle$$

$$= 0 \qquad \square$$

Projection Theorem

Let W be a finite dimensional subspace of inner product space V. Then every $v \in V$ can be expressed as a sum

$$v = w + w^{\perp}$$

where $w \in W$, $w^{\perp} \in W^{\perp}$.

Pf: Let $S = \{v_1, \ldots v_n\}$ be an orthonormal basis of W. By previous discussion

$$w = \text{proj}_W v \quad \in W$$

$$w^{\perp} = v - w \quad \in W^{\perp} \qquad \square$$

Normalized versus non normalized

It is more common to work with non-normalized (orthogonal) vectors.

Suppose $S = \{v_1, \ldots v_n\}$ is not normalized. Then

$$\text{proj}_W v = \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \cdots \frac{\langle v, v_n \rangle}{\langle v_n, v_n \rangle} v_n$$

versus

$$\text{proj}_W v = \langle v, \hat{v}_1 \rangle \hat{v}_1 + \cdots \langle v, \hat{v}_n \rangle \hat{v}_n$$

if the $\hat{v}_k$ are normalized, $\|\hat{v}_k\| = 1$.

EXAMPLE $V = \mathbb{R}^4$, $W = \text{span}\{v_1\} = \text{span}\{(1,1,0,-2)\}$

Find the orthogonal decomposition of $v = (2,1,3,4)$ relative to W:

(1) $\qquad v = w + w^{\perp} \qquad\qquad w \in W \qquad w^{\perp} = W^{\perp}$

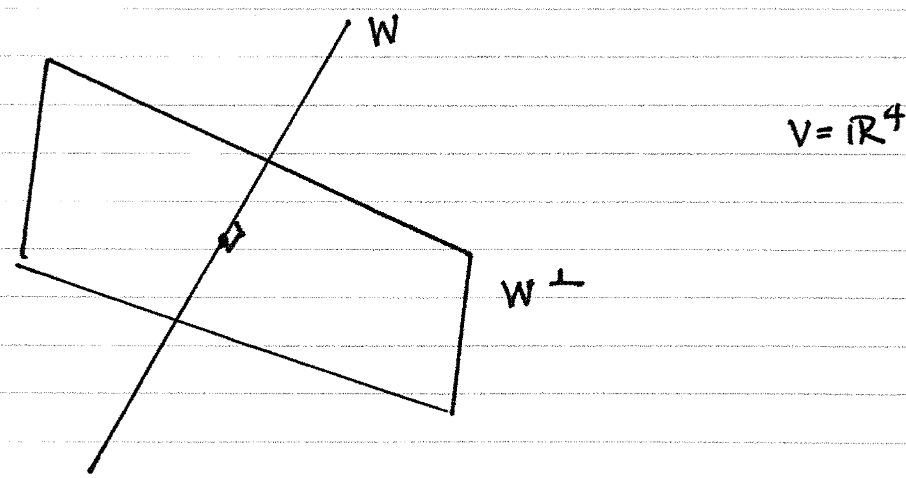
First we find w

$$w = \text{proj}_W v = \frac{\langle v, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = -\frac{5}{6} v_1 = -\frac{5}{6}(1,1,0,-2)$$

Then from (1)

$$w^{\perp} = v - w = \frac{1}{6}(17, 11, 18, 14)$$

One can verify $\langle w, w^{\perp}\rangle = 0$.



EXAMPLE $V = M_{22}$, $\langle u, v \rangle \equiv \mathrm{Tr}(u^T v)$

$$W = \mathrm{span}\{v_1\} = \mathrm{span}\left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right\} \qquad v \equiv \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

Pre calculate inner products for projection

$$\langle v, v_1 \rangle = 4$$

$$\langle v_1, v_1 \rangle = 3$$

Thus

$$w = \mathrm{proj}_W v = \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = \frac{4}{3} v_1$$

To summarize

$$w = \begin{bmatrix} 4/3 & 4/3 \\ 0 & 4/3 \end{bmatrix}$$

$$w^{\perp} = v - w = \begin{bmatrix} 2/3 & -1/3 \\ -1 & -1/3 \end{bmatrix}$$

EXAMPLE $\quad V = M_{22} \qquad \langle u, v\rangle = Tr(u^T v)$

Given W in terms of orthogonal basis:

$$W = span\{w_1, w_2\} = span\left\{\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 1 & -2 \end{bmatrix}\right\}$$

Easy to check $\langle w_1, w_2\rangle = 0$. Let

$$v = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

Compute projection of v onto W

(1) $\quad w = proj_W = \dfrac{\langle v, w_1\rangle}{\langle w_1 w_1\rangle} w_1 + \dfrac{\langle v, w_2\rangle}{\langle w_2 w_2\rangle} w_2$

$$= \frac{6}{6} w_1 + \frac{(-2)}{9} w_2$$

$$= w_1 - 2/9\, w_2$$

$$= \begin{bmatrix} 13/9 & 2 \\ -2/9 & -5/9 \end{bmatrix}$$

Then the orthogonal decomposition is

$$v = w + w^\perp$$

where $w^\perp$ can be computed (knowing (1))

$$w^\perp = v - w$$

EXAMPLE $\quad V = \mathbb{R}^3 \qquad W = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$

To carry out a projection calculation we need an orthogonal basis. In general we use Gram-Schmidt process (next section) to find one. We postpone this discussion until later and note:

$$W = \text{span}\{w_1, w_2\} = \text{span}\left\{\frac{1}{\sqrt{2}}(1,0,-1), \frac{1}{\sqrt{6}}(1,-2,1)\right\}$$

Can easily verify $\langle w_1, w_2 \rangle = 0$ and $w_i \in W$, $\underline{\|w_i\| = 1}$.

For $v = (1,1,-1)$ we can compute

(1) $\quad w = \text{proj}_W v = \langle v, w_1 \rangle w_1 + \langle v, w_2 \rangle w_2$

$$= \sqrt{2}\, w_1 - \tfrac{1}{3}\sqrt{6}\, w_2$$

$$= \left(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{4}{3}\right)$$

Then the orthogonal decomposition

$$v = w + w^{\perp}$$

is completed by calculating

(2) $\quad w^{\perp} = v - w = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$

It is easily verified $\langle w, w^{\perp} \rangle = 0$.

Projections in $\mathbb{R}^n$ as Linear Transformations

Let $V = \mathbb{R}^n$ and $S = \{w_1, \ldots w_n\}$ be an orthogonal basis (in V) for

$$W = \text{span}\{w_1, \ldots w_n\}$$

Then for any $v \in V$

$$\boxed{T(v) = \text{proj}_W v = \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \cdots \frac{\langle v, v_n \rangle}{\langle v_n, v_n \rangle} v_n}$$

It is readily verified

(1) $T : \mathbb{R}^n \to \mathbb{R}^n$

(2) $T(kv) = kT(v) \quad \forall k \in \mathbb{R}$

(3) $T(v_1 + v_2) = T(v_1) + T(v_2) \quad \forall v_k \in \mathbb{R}^n$

That is to say the operation of projection is linear and since $V = \mathbb{R}^n$ there exists a matrix such that

$$T(v) = Av$$

In particular, if e_k are standard basis,

$$A = \left[T(e_1) \cdots T(e_n) \right]$$

EXAMPLE $V = \mathbb{R}^3$ $\quad W = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$

As discussed before

$$W = \text{span}\{v_1, v_2\} \qquad v_1 = \tfrac{1}{\sqrt{2}}(1, 0, -1) \quad v_2 = \tfrac{1}{\sqrt{6}}(1, -2, 1)$$

Then $T : \mathbb{R}^3 \to \mathbb{R}^3$ projection operator is

$$T(v) = \text{proj}_W v = \langle v, v_1 \rangle v_1 + \langle v, v_2 \rangle v_2$$

where $T(v) = Av$, $A = [T(e_1)\ T(e_2)\ T(e_3)]$.

$$T(e_1) = \text{proj}_W e_1 = \tfrac{1}{\sqrt{2}} v_1 + \tfrac{1}{\sqrt{6}} v_2 = \left(\tfrac{2}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}\right)$$

$$T(e_2) = \text{proj}_W e_2 = 0 v_1 - \tfrac{\sqrt{6}}{3} v_2 = \left(-\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right)$$

$$T(e_3) = \text{proj}_W e_3 = -\tfrac{1}{\sqrt{2}} v_1 + \tfrac{1}{\sqrt{6}} v_2 = \left(-\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{2}{3}\right)$$

Compiling the results

$$T(v) = Av = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} v$$

<u>Remark</u>: If one used a different basis (orthog) for W the resulting A would be the same. Hard to prove.

Gram-Schmidt Orthogonalization

Every finite dimensional inner product space has an orthonormal basis.

$S = \{u_1, u_2, .. u_n\}$ basis

$S_\perp = \{v_1, v_2, ... v_n\}$ orthogonal basis.

Given S the basis $S_\perp$ is found using the following Algorithm

(1) $v_1 = u_1$ $\qquad W_1 = \text{span}\{v_1\}$

(2) $v_2 = u_2 - \text{proj}_{W_1} u_2$ $\qquad W_2 = \text{span}\{v_1, v_2\}$

(3) $v_K = u_K - \text{proj}_{W_{K-1}} u_K$ $\qquad W_K = \text{span}\{v_1, .. v_K\}$

Proof : Inductive (Text)

EXAMPLE $\quad S = \{1, x, x^2\} \quad V = P_2 \quad \langle u, v\rangle = \int_0^1 u(x)v(x)dx$

$$v_1 = u_1 = 1$$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = u_2 - \frac{1/2}{1} v_1 = x - \tfrac{1}{2}$$

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 = \cdots = x^2 - x + \tfrac{1}{6}$$

Thus

$$S_\perp = \{1, \ x - \tfrac{1}{2}, \ x^2 - x + \tfrac{1}{6}\}$$