EXAMPLE Use GS-Algorithm to convert the basis S of $V = \mathbb{R}^3$ into an orthogonal one $S_\perp$. Given

$$S = \{u_1, u_2, u_3\} = \{(1,1,1), (2,0,-1), (1,-1,1)\}$$

Many calculations are suppressed. Orthogonal basis $S_\perp = \{v_1, v_2, v_3\}$

$$v_1 = u_1$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = u_2 - \frac{1}{3} v_1 = \left(\frac{5}{3}, -\frac{1}{3}, \frac{4}{3}\right)$$

Lastly

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$v_3 = u_3 - \frac{2}{7} v_1 - \frac{1}{7} v_2$$

$$v_3 = \frac{1}{7}(3, -9, 6)$$

Summary (wlog)

$$S_\perp = \{(1,1,1), (5,-1,4), (3,-9,6)\}$$

EXAMPLE $\quad V = P_2 \qquad \langle u, v\rangle = \int_{-1}^{1} u(x)v(x)\,dx$

Find $\text{proj}_W v$ where $v = x^2 + x + 1$ and

$$W = \text{span}\{u_1, u_2\} = \text{span}\{x+1, 4x^2\}$$

Problem: The basis for W is not orthogonal. Can check $\langle u_1, u_2\rangle \neq 0$. So, before we compute $\text{proj}_W v$, we find an orthogonal basis

$$S_\perp = \{w_1, w_2\} \qquad \langle w_1, w_2\rangle = 0$$

for W using Gram-Schmidt.

$$w_1 = u_1 \qquad = x + 1$$

$$w_2 = u_2 - \frac{\langle u_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 \qquad = 4x^2 - x - 1$$

Then

$$\text{proj}_W v = \frac{\langle v, w_1\rangle}{\langle w_1, w_1\rangle} w_1 + \frac{\langle v, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$

$$= \tfrac{5}{4} w_1 + \tfrac{1}{4} w_2 = x^2 + x + 1 \quad = v \;!!$$

So $v = w + w^\perp$ where $w^\perp = 0$.