Projections in $\mathbb{R}^n$ as Linear Transformations

Let $V = \mathbb{R}^n$ and $S = \{w_1, \ldots w_n\}$ be an orthogonal basis (in V) for

$$W = \text{span}\{w_1, \ldots w_n\}$$

Then for any $v \in V$

$$\boxed{T(v) = \text{proj}_W v = \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \cdots \frac{\langle v, v_n \rangle}{\langle v_n, v_n \rangle} v_n}$$

It is readily verified

(1) $T : \mathbb{R}^n \to \mathbb{R}^n$

(2) $T(kv) = k\,T(v) \qquad \forall k \in \mathbb{R}$

(3) $T(v_1 + v_2) = T(v_1) + T(v_2) \qquad \forall v_k \in \mathbb{R}^n$

That is to say the operation of projection is linear and since $V = \mathbb{R}^n$ there exists a matrix such that

$$T(v) = A v$$

In particular, if e_k are standard basis,

$$A = \left[T(e_1) \cdots T(e_n) \right]$$

EXAMPLE $V = \mathbb{R}^3$ $\quad W = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$

As discussed before

$$W = \operatorname{span}\{v_1, v_2\} \qquad v_1 = \tfrac{1}{\sqrt{2}}(1,0,-1) \quad v_2 = \tfrac{1}{\sqrt{6}}(1,-2,1)$$

Then $T : \mathbb{R}^3 \to \mathbb{R}^3$ projection operator is

$$T(v) = \operatorname{proj}_W v = \langle v, v_1 \rangle v_1 + \langle v, v_2 \rangle v_2$$

where $T(v) = Av$, $A = [T(e_1)\ T(e_2)\ T(e_3)]$.

$$T(e_1) = \operatorname{proj}_W e_1 = \tfrac{1}{\sqrt{2}} v_1 + \tfrac{1}{\sqrt{6}} v_2 = \left(\tfrac{2}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}\right)$$

$$T(e_2) = \operatorname{proj}_W e_2 = 0\, v_1 - \tfrac{\sqrt{6}}{3} v_2 = \left(-\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right)$$

$$T(e_3) = \operatorname{proj}_W e_3 = -\tfrac{1}{\sqrt{2}} v_1 + \tfrac{1}{\sqrt{6}} v_2 = \left(-\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{2}{3}\right)$$

Compiling the results

$$T(v) = Av = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} v$$

<u>Remark</u>: If one used a different basis (orthog) for W the resulting A would be the same. Hard to prove.

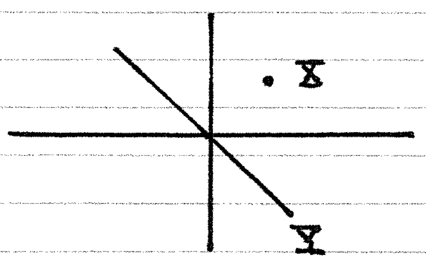
Sums and Direct Sums

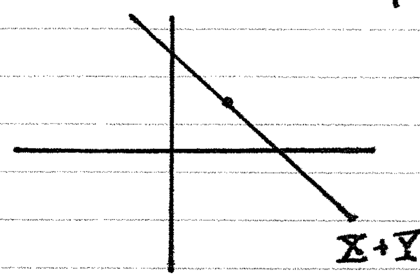
In the following V is a vector space and X, Y are subsets of V

Defn: $X, Y \subset V$. We define the sum

$$X + Y = \{ z : z = x + y \ ,\ x \in X,\ y \in Y \}$$

EXAMPLE $V = \mathbb{R}^2$ $\quad X = \{(1,1)\}$ $\quad Y = \text{span}\{(1,-1)\}$





note in this case $X+Y$ is not a subspace of V since it doesn't contain $\vec{0}$.

EXAMPLE $V = \mathbb{R}^2$ $\quad X = \text{span}\{(1,1)\}$ $\quad Y = \text{span}\{(1,-1)\}$

$$X + Y = \mathbb{R}^2$$

EXAMPLE $V = P_2$ $\quad X_1 = \text{span}\{1\},\ X_2 = \text{span}\{x\}$

$X_3 = \text{span}\{x + x^2\},\ X_4 = \text{span}\{x^2 - 4\}$

$$X_1 + X_2 + X_3 + X_4 = P_2$$

Defn **(Direct Sum)**

Let V be a vector space

(1) X, Y be subspaces of V

(2) $X \cap Y = \{0\}$ (disjoint)

(3) $V = X + Y$

then V is a direct sum of X, Y and is written

$$V = X \oplus Y$$

EXAMPLE (orthogonal complements)

$$V = W \oplus W^{\perp}$$

$$\mathbb{R}^n = \text{row } A \oplus N(A)$$

$$\mathbb{R}^m = \text{Col } A \oplus N(A^T)$$

EXAMPLE $X = \text{span}\{(1,1,0), (1,-1,0)\}$

$Y = \text{span}\{(1,0,0), (0,0,1)\}$

$\mathbb{R}^3 = X + Y$ but $X \cap Y \neq \{0\} \Rightarrow \mathbb{R}^3 \neq X \oplus Y$

EXAMPLE $A = A^T$ with real dist. evals $\lambda_1, \dots \lambda_n$.

$$\mathbb{R}^n = \bigoplus_{i=1}^{n} E_{\lambda_i}(A)$$