General Linear Transformations.

Defn: A mapping $T : \mathbb{X} \to \mathbb{Y}$ is linear iff

(a) $T(x+y) = T(x) + T(y) \quad \forall x, y \in \mathbb{X}$

(b) $T(kx) = kT(x) \quad \forall x \in \mathbb{X},\ k \in \mathbb{R}$

To date our key example of a linear transformation has been matrix transformations

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$T_A(x) = Ax \qquad A \in \mathbb{R}^{m \times n}$$

Two key spaces associated with T_A are

$$N(A) \qquad col(A)$$

or the nullspace of A and the column space of A. The column space is exactly the same as the range $R(T_A)$ of T_A. Generalizations:

Defn: Let $T : \mathbb{X} \to \mathbb{Y}$ be linear. Then we define the kernel and range of T:

$$\ker(T) = \{x \in \mathbb{X} : T(x) = 0\}$$

$$R(T) = \{y \in \mathbb{Y} : \exists x \in \mathbb{X},\ T(x) = y\}$$

Some authors use $N(T)$ for $\ker(T)$.

EXAMPLE **Matrix transformations**

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$T(x) \equiv Ax \qquad A \in \mathbb{R}^{m \times n}$$

$$\ker(T) = N(A) \qquad \text{kernel}$$

$$R(T) = \mathrm{col}(A) \qquad \text{range}$$

EXAMPLE **Projections**

$$T(x) \equiv \mathrm{proj}_W x \qquad W \text{ subspace } X$$

$$T : X \to W$$

Let $S_\perp = \{v_1, v_2, \dots v_n\}$ be an orthonormal basis for W

(1) $\mathrm{proj}_W x = \langle x, v_1 \rangle v_1 + \cdots \langle x, v_n \rangle v_n = 0$

What x satisfy (1), i.e. $T(x) = 0$. Taking inner product of (1) with v_k and using $\langle v_i, v_k \rangle = 0$ if $i \neq k$ we conclude

$$\langle x, v_k \rangle = 0 \qquad k = 1, 2 \dots n$$

concluding

$$\ker(T) = W^\perp$$

$$R(T) = W$$

EXAMPLE — Coordinates

Let $S = \{v_1, \ldots v_n\}$ be a basis for vector space $\underline{X}$. Then

$$T(x) \equiv (x)_S$$

$$T : \underline{X} \rightarrow \mathbb{R}^n$$

is a linear transformation.

Linear independence of $v_k \Rightarrow T(x) = 0$ only if $x = 0$.

$$\ker(T) = \{0\}$$

$$R(T) = \mathbb{R}^n$$

EXAMPLE — Multiplication by x

Let $\underline{X} = P_n$ and $\underline{Y} = P_{n+1}$. Define

$$T(u) = x\,u(x) \qquad u \in P_n$$

$$T : \underline{X} \rightarrow \underline{Y}$$

Since $x\,u(x) = 0 \;\forall x$ iff $u(x) \equiv 0$ we have

$$\ker(T) = \{0\}$$

Also

$$R(T) = P_{n+1} \setminus \text{span}\{1\}$$

take away

EXAMPLE Inner Products

Let $\mathbb{X}$ be an inner product space. Let $x_0 \in \mathbb{X}$

$$T(x) \equiv \langle x, x_0 \rangle$$

$$T : \mathbb{X} \to \mathbb{R}$$

Let $W = \text{span}\{x_0\}$. Then $x \in \ker(T) \Rightarrow \langle x, x_0 \rangle = 0$.
Thus

$$\ker(T) = W^{\perp}$$

$$R(T) = \mathbb{R}$$

EXAMPLE Differentiation

Let $\mathbb{X} = C^1(\mathbb{R})$ and $\mathbb{Y} = C(\mathbb{R})$.

$$T(u) \equiv \frac{du}{dx}$$

$$T : \mathbb{X} \to \mathbb{Y}$$

Here $\mathbb{X}, \mathbb{Y}$ are not finite dimensional vector spaces. Still

$$\ker(T) = \text{span}\{1\} \qquad \text{constant fns.}$$

It should be evident that $R(T) = \mathbb{Y}$, i.e., all of $\mathbb{Y}$.
To be clear, if $y \in \mathbb{Y}$ then if

$$u(x) = \int_0^x y(t)\,dt$$

exists and $u' = y(x)$.

<u>**EXAMPLE**</u> <u>Integration.</u>

Let $\mathbb{X} = C(\mathbb{R})$ and $\mathbb{Y} = C^1(\mathbb{R})$. Then

$$T(u) \equiv \int_0^x u(t)\,dt$$

is a linear map $T : \mathbb{X} \to \mathbb{Y}$. The kernel $\ker(T)$ consists of all $u \in \mathbb{X}$ such that

$$\int_0^x u(t)\,dt = 0 \qquad \forall x$$

Differentiating this in $x \Rightarrow u(x) = 0, \forall x$.

$$\ker(T) = \{0\}$$

<u>**EXAMPLE**</u> <u>Translation</u>

Let $\mathbb{X}$ be any vector space and $x_0 \in \mathbb{X}$

$$T(x) \equiv x + x_0 \qquad x_0 \neq 0$$

is not a linear map. Can see this from

$$T(x+y) \neq T(x) + T(y)$$

$$x + y + x_0 \neq x + y + 2x_0$$

Also $T(kx) = kx + x_0 \neq kT(x) = kx + kx_0$.

EXAMPLE Evaluation map

Let $\underline{X} = C(\mathbb{R})$. Then $T : \underline{X} \to \mathbb{R}$ defined by

$$T(u) \equiv u(1) \qquad u(x) \in C(\mathbb{R})$$

is a linear map.

$$\ker(T) = \{ u \in \underline{X} : u(1) = 0 \}$$

$$R(T) = \mathbb{R}$$

EXAMPLE More exotic

Let $\underline{X} = P_n$ and $u \in \underline{X} \Rightarrow u(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$

$$T(u) \equiv (a_n, a_{n-1})$$

$$T : P_n \to \mathbb{R}^2$$

is linear. Also, $\ker(T) = P_{n-2}$, $R(T) = \mathbb{R}^2$

EXAMPLE Maximum degree

Let $\underline{X} = P_n$ and define $T : P_n \to \mathbb{Z}$ where

$$T(u) = \text{maximum degree of polynomial } u$$

For instance, $T(u) = 3$ if $u = x^3 + x^2 + x$.

Clearly not linear. Take any two $u_1, u_2 \in \underline{X}$ with $T(u_k) = n$.

$$T(u_1) + T(u_2) = 2n > n \geq T(u_1 + u_2)$$