Images from basis vectors

Let $S = \{u_1, \ldots u_n\}$ be a basis for X and suppose $T: X \to Y$ is a linear map. Then

$$T(u) = c_1 T(u_1) + \cdots c_n T(u_n) \qquad c_k \in \mathbb{R}$$

where $(u)_S = (c_1, \ldots c_n)$. Moreover, $R(T) \leq n$ and

$$R(T) = \text{span}\{T(u_1), \ldots T(u_n)\}$$

__EXAMPLE__ Let $S = \{u_1, u_2, u_3\}$ be a basis for X and $T: X \to \mathbb{R}^3$. Given

$$T(u_1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad T(u_2) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad T(u_3) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

For any u we can compute $T(u)$. Suppose $(u)_S = (2, 3, -1)$

$$T(u) = 2T(u_1) + 3T(u_2) - T(u_3) = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

To find the range $R(T)$ note it is the col(A) where A has $T(u_k)$ as columns

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

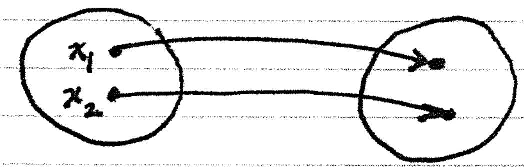
hence $\dim R(T) = 2$ and

$$R(T) = \text{span}\{T(u_1), T(u_2)\}$$

Inverse Transformations

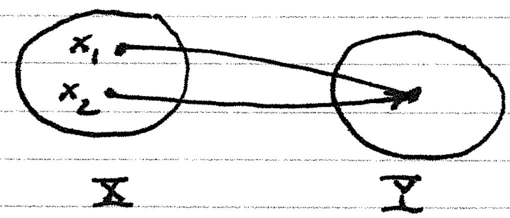
Defn: Let $T: \mathbb{X} \to \mathbb{Y}$ be a linear transformation. We say T is 1-1 if for every $y \in R(T)$ there exists a unique $x \in \mathbb{X}$ s.t. $T(x) = y$.

1-1



$T(x_1) \neq T(x_2)$

not 1-1



$T(x_1) = T(x_2)$
for some $x_1 \neq x_2$

EXAMPLE Let $\mathbb{X} = P_2$ and $\mathbb{Y} = P_2$ and define

$$T(u) = a_1 x + a_0 \qquad u(x) = a_2 x^2 + a_1 x + a_0$$

T is linear but not 1-1. As an explicit example $T(u_k) = 2x$ for $u_1 = x^2 + 2x$, $u_2 = 17x^2 + 2x$.

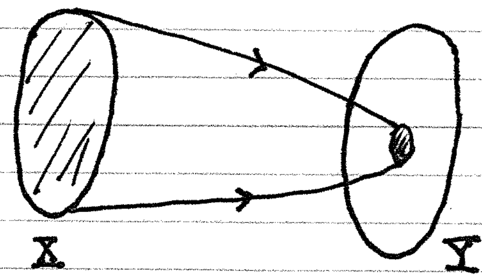
Remark on Inverse

If $T: \mathbb{X} \to \mathbb{Y}$ is linear and 1-1 it has an inverse.

Recall $\forall y \in R(T)$ there exists a unique $x \in \mathbb{X}$, $T(x) = y$. This uniqueness defines a function from $\mathbb{Y}$ to $\mathbb{X}$

$$T^{-1}: R(T) \to \mathbb{X} \qquad T^{-1}(T(x)) = x$$

Defn Let $T: X \to Y$ be a linear transformation. We say T is <u>onto</u> if $T(X) = Y$



not <u>onto</u>

Theorem Let X and Y be finite dimensional vector spaces and $T: X \to Y$ is linear. Then, the following are equivalent

(i) T is 1-1

(ii) $\ker(T) = \{0\}$

(iii) $R(T) = Y$ (onto)

Pf (i) $\Rightarrow$ (ii) Since T is linear, $T(0) = 0$. Since T is 1-1 $\nexists x \neq 0$ s.t. $T(x) = 0$ too. Thus, $\ker(T) = \{0\}$

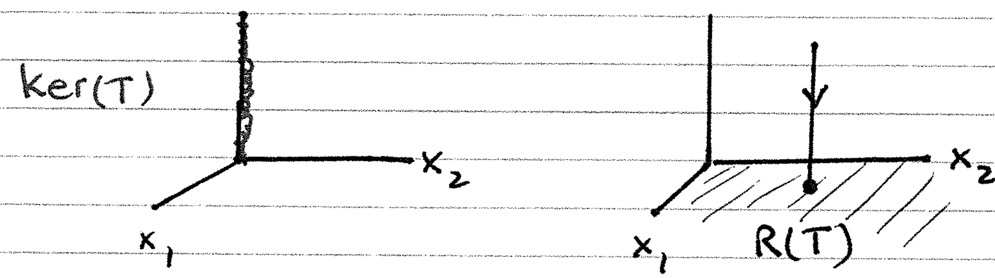
(ii) $\Rightarrow$ (i) Assume $\ker(T) = \{0\}$. Let $y \in Y$ and $T(x_1) = T(x_2) = y$. By linearity

$$T(x_1 - x_2) = 0$$

and hence $x_1 = x_2$ showing T is 1-1.

EXAMPLE Orthogonal Projection

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x) = (x_1, x_2, 0)$.



$\ker(T) \neq \{0\}$ hence T not 1-1 (no inverse)

EXAMPLE Differentiation

Let $D: C'(\mathbb{R}) \to C(\mathbb{R})$ and $D(f) \equiv \frac{df}{dx}$. (is linear!)

$$\ker(D) = \{u : u(x) \equiv k \text{ for some } k \in \mathbb{R}\}$$

$$\ker(D) \neq \{0\}$$

hence D is not 1-1 (nonunique inverse)

EXAMPLE Integration

Let $T: C(\mathbb{R}) \to C'(\mathbb{R})$ and $T(f) \equiv \int_0^x f(t)\,dt$

$f \in \ker(T)$ iff

$$0 = \int_0^x f(t)\,dt \qquad \forall x$$

differentiatiate in x gives $f(x) \equiv 0 \Rightarrow \ker(T) = \{0\}$.

EXAMPLE Rotation $T_\theta(x)$, $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

$$T_\theta(x) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ has } \ker(T_\theta) = 0, \quad T_\theta^{-1} = T_{-\theta}$$

Defn: A linear map $T: V \to W$ that is 1-1 and onto is said to be an isomorphism and W is isomorphic to V.

In what follows we shall use the unproven fact

$$T : \mathbf{X} \to \mathbf{Y} \qquad \text{linear 1-1}$$

$$T^{-1} : \mathbf{Y} \to \mathbf{X} \qquad \text{linear 1-1}$$

EXAMPLE $T : P_2 \to \mathbb{R}^3$ defined by

$$T(c_0 + c_1 x + c_2 x^2) = (c_0, c_1, c_2)$$

$$c_0 + c_1 x + c_2 x^2 \underset{T^{-1}}{\overset{T}{\rightleftarrows}} (c_0, c_1, c_2)$$

Then the images of $T(ku)$ and $T(u_1 + u_2)$ in a 1-1 correspondence

$$k(c_0 + c_1 x + c_2 x^2) \overset{T}{\rightleftarrows} k(c_0, c_1, c_2)$$

$$kc_0 + kc_1 x + kc_2 x^2 \overset{T}{\rightleftarrows} (kc_0, kc_1, kc_2)$$

and similar for addition

$$(c_0 + c_1 x + c_2 x^2) + (d_0 + d_1 x + d_2 x^2) \overset{T}{\rightleftarrows} (c_0, c_1, c_2) + (d_0, d_1, d_2)$$

$$(c_0 + d_0) + (c_1 + d_1)x + (c_2 + d_2)x^2 \overset{T}{\rightleftarrows} (c_0 + d_0, c_1 + d_1, c_2 + d_2)$$

Theorem: Every real n-dimension vector space is isomorphic to $\mathbb{R}^n$

Pf/ Let V be a vector space with (any) basis

$$S \equiv \{v_1, \dots v_n\}$$

Define

$$T(u) \equiv (u)_S = (c_1, c_2 \dots c_n)$$

Suffices to show $T: V \to \mathbb{R}^n$ is 1-1 and onto. We previously showed T is linear. Let

$$T(v) = (v)_S = (d_1, d_2, \dots d_n)$$

If $u \neq v$ then at least one coordinate differs, i.e. $c_i \neq d_i$ for some i. Hence

$$u \neq v \quad \Rightarrow \quad T(u) \neq T(v) \qquad \text{1-1}$$

Clearly T is onto. Let $c = (c_1, \dots c_n) \in \mathbb{R}^n$ be any $c \in \mathbb{R}^n$. If

$$u = c_1 v_1 + \cdots c_n v_n$$

then $T(u) = c$. Every $c \in \mathbb{R}^n$ has a (unique) preimage.

Ex Isomorphism between P_n and $\mathbb{R}^{n+1}$

$$a_0 + a_1 x + \cdots a_n x^n \xrightarrow{T} (a_0, a_1, \ldots a_n)$$

Ex Isomorphism between M_{22} and $\mathbb{R}^4$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{T} (a, b, c, d)$$

Ex Differentiation by Matrix Multiplication

Consider differention on P_3 as in $D: P_3 \to P_2$

$$a_0 + a_1 x + \cdots a_3 x^3 \xrightarrow[D(u)]{} a_1 + 2a_2 x + 3a_3 x^2$$

Relative to basis $E = \{1, x, \ldots x^3\}$

$$(u)_E = (a_0, a_1, \ldots a_3)$$

$$(D(u))_E = (a_1, 2a_2, 3a_3, 0)$$

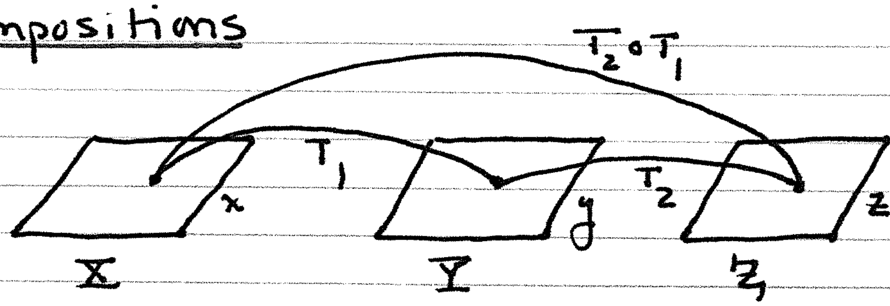
So that

$$A\,(u)_E = (T(u))_E$$

is given explicitly by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{bmatrix}$$

Compositions



Theorem: If $T_1 : \mathbb{X} \to \mathbb{Y}$ and $T_2 : \mathbb{Y} \to \mathbb{Z}$ are linear then $(T_2 \circ T_1) : \mathbb{X} \to \mathbb{Z}$ is linear.

Pf/

(a) $$(T_2 \circ T_1)(x_1 + x_2) = T_2(T_1(x_1 + x_2))$$
$$= T_2(T_1(x_1) + T_1(x_2))$$
$$= T_2(T_1(x_1)) + T_2(T_1(x_2))$$
$$= (T_2 \circ T_1)(x_1) + (T_2 \circ T_1)(x_2)$$

(b) $$(T_2 \circ T_1)(kx) = k\, T_2(T_1(x)) = k(T_2 \circ T_1)(x) \quad \square$$

EXAMPLE

$T_1 : P_1 \to P_2$ $\qquad T_1(u) \equiv x\, u(x)$

$T_2 : P_2 \to P_2$ $\qquad T_2(u) \equiv u(x-1)$ $\quad$ linear!

$(T_2 \circ T_1) : P_1 \to P_2$ $\qquad (T_2 \circ T_1)(u) = (x-1)\, u(x-1)$

For example if $u(x) = 3x+1$ then

$$T_1(u) = x(3x+1)$$

$$T_2(T_1(u)) = (x-1)(3(x-1)+1)$$

Example **Noncommuting compositions.**

$$T_1(x,y) = (2x, 3y)$$

$$T_2(x,y) = (x-y, x+y)$$

Then

$$T_2(T_1(x,y)) = T_2(2x, 3y)$$

$$= (2x-3y, 2x+3y)$$

$$T_1(T_2(x,y)) = T_1(x-y, x+y)$$

$$= (2x-2y, 3x+3y)$$

Comparing

$$(T_2 \circ T_1)(x,y) \neq (T_1 \circ T_2)(x,y)$$

Remarks As matrix transformations

$$T_1(\vec{x}) = A_1 \vec{x} \qquad A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$T_2(\vec{x}) = A_2 \vec{x} \qquad A_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Easy to verify

$$A_1 A_2 \neq A_2 A_1$$