Algebraic Properties of Matrices.

In the following $A, B, C \in \mathbb{R}^{n \times n}$ and $\alpha \in \mathbb{R}$

$A + B = B + A$ commutative addition

$A + (B + C) = (A + B) + C$ associative addition

$A(BC) = (AB)C$ associative multiplicat.

$A(B + C) = AB + AC$ left distributive law

$(B + C)A = BA + CA$ right distributive law

$\alpha(A + B) = \alpha A + \alpha B$

$\alpha(AB) = (\alpha A)B = A(\alpha B)$

Recall the important exception that for most A and B

$AB \neq BA$ not commutative

EXAMPLE $(A + B)^2 = (A + B)(A + B)$

$$= A^2 + AB + BA + B^2$$

EXAMPLE Polynomial matrices

$P_n(x) = a_n x^n$ repeated index

$$P_n(A) \equiv a_n A^n + a_{n-1} A^{n-1} + \cdots a_0 I$$

Inverse of a square matrix

Defn: Let $A, B \in \mathbb{R}^{n \times n}$ and suppose

$$AB = BA = I$$

Then B is the inverse of A and we write

$$B = A^{-1}$$

EXAMPLE $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

Can easily show $AB = BA = I$ so $B = A^{-1}$

EXAMPLE A matrix with a column of zeros can't have an inverse

$$A = \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{array}\right] \quad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Clearly there are many nonzero c s.t. $Ac = 0$
For instance $c = (0, 0, \alpha)^T$ and any $\alpha \in \mathbb{R}$.

But, if A had an inverse

$$A^{-1}Ac = A^{-1}0$$

$$Ic = 0 \quad \Rightarrow c = 0$$

This is a contradiction.

Inverse of $A \in \mathbb{R}^{2\times 2}$

Let $A \in \mathbb{R}^{2\times 2}$ and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Define the determinant of A by

$$\det A \equiv ad - bc$$

Then, if $\det A \neq 0$ the inverse of A is

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

EXAMPLE Find the solution of $Ax = b$ given

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \qquad b = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Note $x = A^{-1}b$ and $\det A = -1 \neq 0 \Rightarrow$

$$A^{-1} = \frac{1}{(-1)} \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$$

Then

$$x = A^{-1}b = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$$

Left versus right inverses

Recall that B is the inverse of A if

$$AB = BA = I$$

Implicit is the fact both matrices are square $A, B \in \mathbb{R}^{n \times n}$.

One might ask if $AB = I$ isn't it automatically true $BA = I$?

Suppose $AB = I$ and A is invertible. Then $\det A \neq 0$ and

$$\det AB = \det A \det B = 1$$

implies $\det B \neq 0$ so that B^{-1} exists. Then

$$AB = I$$

$$B(AB) = B$$

$$(BA)B = B$$

$$(BA)BB^{-1} = BB^{-1}$$

$$BA = I \qquad \square$$

Theorem Let $A, B \in \mathbb{R}^{n \times n}$

$$BA = I \Rightarrow B = A^{-1}$$

$$AB = I \Rightarrow B = A^{-1}$$

Algebraic Properties of Inverse and Transpose

In the following $A, B \in \mathbb{R}^{n \times n}$ are invertible and $\alpha \in \mathbb{R}$ is a real number

(i) $(AB)^{-1} = B^{-1}A^{-1}$

(ii) $(A^{-1})^{-1} = A$

(iii) $(A^n)^{-1} = (A^{-1})^n$

(iv) $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$ $\alpha \in \mathbb{R}$

(v) $(A^T)^{-1} = (A^{-1})^T$

<u>Proof of (i)</u> First show $X X^{-1} = I$ where $X = AB$

$$\begin{aligned}(AB)(AB)^{-1} &= (AB)(B^{-1}A^{-1}) \\ &= A(BB^{-1})A^{-1} \\ &= A \quad I \quad A^{-1} \\ &= I\end{aligned}$$

Likewise show $\underline{X^{-1}}\,\underline{X} = \underline{I}$ (redundant)

$$\begin{aligned}(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\ &= B^{-1} \quad I \quad B \\ &= I\end{aligned}$$

<u>Proof of (ii)</u> Note $AXX^{-1} = A$. Here $X = A^{-1}$

$$\begin{aligned}A = A A^{-1}(A^{-1})^{-1} &= (AA^{-1})(A^{-1})^{-1} \\ &= I \quad (A^{-1})^{-1} \\ &= (A^{-1})^{-1}\end{aligned}$$

□

Pf of (iii) Induction. True for n=1

$$\underbrace{(A \cdot A \cdots A)}_{n\text{-terms}}{}^{-1} = \underbrace{A^{-1} \cdots A^{-1}}_{n\text{-terms}}$$

Pf of (iv) Almost immediate

$$(\alpha^{-1} A^{-1})(\alpha A) = \alpha^{-1} \alpha A^{-1} A = A^{-1} A = \underline{I}$$

Pf of (v) Claim $(A^T)^{-1} = (A^{-1})^T$

$$\begin{aligned} (A^{-1})^T A^T &= (A A^{-1})^T && \text{rules of transpose} \\ &= I^T \\ &= I \end{aligned}$$

Computing A^{-1} using Gaussian elimination

To find A^{-1} one uses row operations to reduce $[A \mid I]$ to reduced upper echelon form, i.e.,

$$[A \,\vdots\, I] \sim [I \,\vdots\, A^{-1}]$$

EXAMPLE (With some steps skipped)

$$\left[\begin{array}{c|c} \underbrace{\begin{matrix}1 & 2 & 3\\ 0 & 2 & 5\\ 1 & 1 & 1\end{matrix}}_{A} & \begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0\\ 0 & 2 & 5 & 0 & 1 & 0\\ 0 & 0 & 1 & -2 & 1 & 2\end{array}\right] \begin{matrix}\text{row}\\ \text{echelon}\end{matrix}$$

$$\sim \left[\begin{array}{c|c} \begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix} & \underbrace{\begin{matrix}-3 & 1 & 4\\ 5 & -2 & -5\\ -2 & 1 & 2\end{matrix}}_{A^{-1}}\end{array}\right] \begin{matrix}\text{reduced}\\ \text{row}\\ \text{echelon}\end{matrix}$$

Remark As a reminder A^{-1} need not exist. May A has no inverse!

Elementary Matrices

All row operations can be represented as (invertible) matrix products. For example, if

$$A \sim B$$

there exists an elementary matrix E_K s.t.

$$B = E_K A$$

Generally E_K is the product of other more elementery matrices.

Broadly there are three types

(1) row operations E_K

(2) scaling operations D_K

(3) permutation operations P_K

Row Operations

$$\underset{E_1}{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} - r_1 - \\ - r_2 - \\ - r_3 - \end{bmatrix}} \underset{\sim}{=} \underset{B}{\begin{bmatrix} - r_1 - \\ - r_2 + a r_1 - \\ - r_3 + b r_1 \end{bmatrix}}$$

Pick a, b to simplify column of B.

EX Henceforth (in examples)

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 2 & 5 \\ 6 & 1 & 1 \end{bmatrix}$$

Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \qquad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ +2 & 1 & 0 \\ +3 & 0 & 1 \end{bmatrix}$$

Then

$$E_1 A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 0 & -5 \\ 0 & -2 & -14 \end{bmatrix}$$

Permutation Operations

A permutation matrix P equals I with its columns permuted. For such matrices

(1) $P^2 = I$ idempotent

(2) $P^{-1} = P$

EXAMPLE. For $n=4$, $A \in \mathbb{R}^{n\times n}$,

$$\underset{P}{\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \left[\begin{matrix} 0 \\ 0 \\ \textcircled{1} \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 0 \\ 0 \\ \textcircled{1} \end{matrix} & \begin{matrix} \textcircled{1} \\ 0 \\ 0 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ \textcircled{1} \\ 0 \\ 0 \end{matrix}\right] \end{array}} \underset{A}{\begin{bmatrix} - r_1 - \\ - r_2 - \\ - r_3 - \\ - r_4 - \end{bmatrix}} = \begin{bmatrix} - r_3 - \\ - r_4 - \\ - r_1 - \\ - r_2 - \end{bmatrix} \quad \begin{matrix} r_3 \rightarrow r_1 \\ r_4 \rightarrow r_2 \\ r_1 \rightarrow r_3 \\ r_2 \rightarrow r_4 \end{matrix}$$

<u>EX</u> Continuing with our previous row reduction we'd like to permute r_2, r_3 of E_1A

$$E_1A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 0 & -5 \\ 0 & -2 & -14 \end{bmatrix} \quad \circlearrowright$$

One can verify

$$(1) \qquad P_2E_1A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & -2 & -14 \\ 0 & 0 & -5 \end{bmatrix}$$

if

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Note that P_2E_1A is now in row echelon form.

While the matrix in (1) above is in row echelon form we will now seek a form with ones on the diagonal

Scaling Operations

$$D = \mathrm{diag}(\alpha_1 \ldots \alpha_n) = \begin{bmatrix} \alpha_1 & & O \\ & \alpha_2 & \\ O & & \ddots \; \alpha_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Then for any matrix $A \in \mathbb{R}^{n \times n}$ the rows of DA are α_k times the row of A :

$$DA = \begin{bmatrix} \alpha_1 & & O \\ & \ddots & \\ O & & \alpha_n \end{bmatrix} \begin{bmatrix} - & r_1 & - \\ - & r_2 & - \\ - & r_n & - \end{bmatrix} = \begin{bmatrix} - & \alpha_1 r_1 & - \\ - & \alpha_2 r_2 & - \\ - & \alpha_n r_n & - \end{bmatrix}$$

EXAMPLE Recall from previous two examples

$$A \sim P_2 E_1 A = \begin{bmatrix} \textcircled{2} & 1 & 5 \\ 0 & \textcircled{-2} & -14 \\ 0 & 0 & \textcircled{-5} \end{bmatrix}$$

Given the results above motivates choice

$$D_3 = \mathrm{diag}(\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{5})$$

and

$$(2) \qquad A \sim D_3 P_2 E_1 A = \begin{bmatrix} 1 & 1/2 & 5/2 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{row echelon}$$

Can think of the sole matrix $X = D_3 P_2 E_1$ as that matrix which row reduces A to the form in (2) above.

Invertibility Theorem (Square matrices)

The following are equivalent

(a) A invertible

(b) $Ax=0 \Rightarrow x=0$

(c) Reduced row echelon of A is I

(d) $Ax=b$ has a unique solution $\forall b \in \mathbb{R}^n$

Proof (a) ⇒ (b) Assume A^{-1} exists. Then if $Ax=0$, does $x=0$?

$$\begin{aligned} Ax &= 0 \\ A^{-1}Ax &= 0 \\ Ix &= 0 \\ x &= 0 \end{aligned}$$

Proof (b) ⇒ (c) Assume $Ax=0 \Rightarrow x=0$.

$$[A \mid 0] \sim [I^* \mid 0]$$

If $I^* \neq I$ then $\exists x$ s.t. $Ax \neq 0$ contradicts assumpt.

Proof (c) ⇒ (d) By assuming (c) we are assured

$$[Ax \mid b] \sim [I \mid b^*]$$

for some b^* in which case $Ax=b$ has a unique solution

(d) ⇒ (a) Constructive proof

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{sole one in } i^{th} \text{ row}$$

We assume $Ax = b$ has a unique solution for each b.

$$A x_i = e_i$$

defines n systems each having a unique solution $x_i \in \mathbb{R}^n$.

Then

$$A \underbrace{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}}_{\text{is } A^{-1}} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & | \\ \vdots & \vdots & & | \end{bmatrix} = I$$

$\square$