Invertibility Theorem (Square matrices)

The following are equivalent

(a) A invertible

(b) $Ax=0 \Rightarrow x=0$

(c) Reduced row echelon of A is I

(d) $Ax=b$ has a unique solution $\forall b \in \mathbb{R}^n$

Proof (a) ⇒ (b) Assume A^{-1} exists. Then if $Ax=0$, does $x=0$?

$$
\begin{aligned}
Ax &= 0 \\
A^{-1}Ax &= 0 \\
Ix &= 0 \\
x &= 0
\end{aligned}
$$

Proof (b) ⇒ (c) Assume $Ax=0 \Rightarrow x=0$.

$$[A \mid 0] \sim [I^* \mid 0]$$

If $I^* \neq I$ then $\exists x$ s.t. $Ax \neq 0$ contradicts assumpt.

Proof (c) ⇒ (d) By assuming (c) we are assured

$$[Ax \mid b] \sim [I \mid b^*]$$

for some b^* in which case $Ax=b$ has a unique solution

(d) $\Rightarrow$ (a) Constructive proof

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{sole one in } i^{th} \text{ row}$$

We assume $Ax = b$ has a unique solution for each b.

$$A x_i = e_i$$

defines n systems each having a unique solution $x_i \in \mathbb{R}^n$.

Then

$$A \underbrace{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}}_{\text{is } A^{-1}} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & | \\ \vdots & \vdots & & \\ \vdots & \vdots & & 1 \end{bmatrix} = I$$

$\square$

Theorem $\boxed{AB \text{ invertible} \Rightarrow A \text{ and } B \text{ invertible}}$

Pf. We show B^{-1} exists since $Bx = 0 \Rightarrow x = 0$ and appeal to equivalency Theorem

Let x_0 be any x_0 such that $Bx_0 = 0$. Then

(1) $$(AB)x_0 = A(Bx_0) = A0 = 0$$

By the invertibility of AB this implies $x_0 = 0$. Thus we have

$$Bx_0 = 0 \quad \Rightarrow x_0 = 0$$

so that B^{-1} exists.

Then

(2) $$(AB)B^{-1} = A(BB^{-1}) = AI = A$$

Rewritten

$$A = \underbrace{(AB)\,B^{-1}}_{\text{product of invertible matrices}}$$

A being the product of invertible matrices is therefore invertible

Symmetric Matrices

Defn: $A \in \mathbb{R}^{n \times n}$ symmetric $\Leftrightarrow$ $A = A^T$

It is easy to see that if A, B are symmetric then

i) A^T symmetric

ii) $A + B$ symmetric

iii) kA symmetric $\forall k \in \mathbb{R}$

From this we see the space of symmetric matrices is closed under addition. It is not closed under matrix multiplication.

EXAMPLE A, B symmetric $\not\Rightarrow$ AB symmetric

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}}_{C}$$

Clearly $A = A^T$, $B = B^T$ but $C \neq C^T$

Theorem $\boxed{\text{If } A, B \text{ are symmetric, } AB \text{ is symmetric if and only if } A \text{ and } B \text{ commute}}$

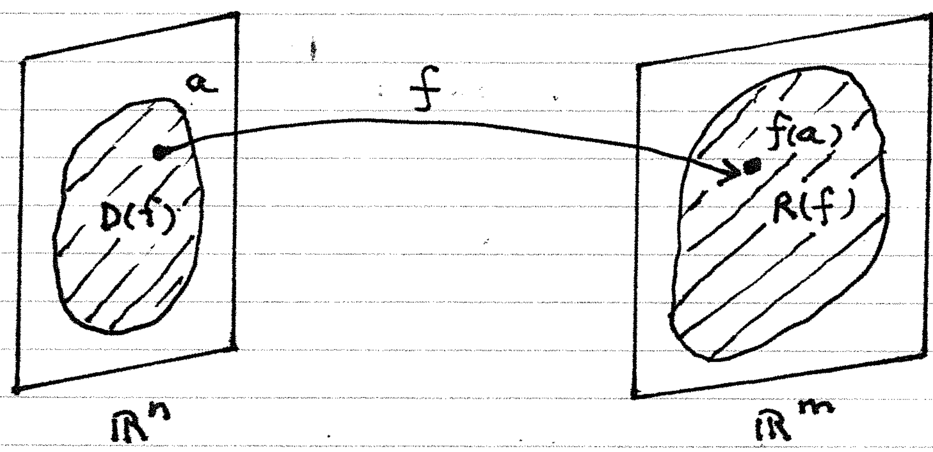
Pf

$$(AB)^T \underset{\text{always}}{=} B^T A^T \underset{\text{by assumption}}{=} (BA)$$

Thus $(AB)^T = AB$ only if $AB = BA$ $\square$

Mappings and Matrix Transformations

A mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ has its domain $D(f) \subset \mathbb{R}^n$. Here f maps into the codomain. The range $R(f)$ of f can be a proper subset of the codomain. Such mappings are also called transformations and $f(a)$ is said to be the image of a under f.

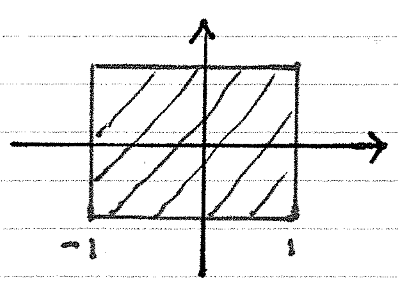


domain $\qquad D(f) = \{ x \in \mathbb{R}^n : f(x) \text{ defined} \}$

range $\qquad R(f) = \{ y \in \mathbb{R}^m : \exists x \in D(f) \text{ with } y = f(x) \}$

EXAMPLE $\quad f : \mathbb{R}^2 \to \mathbb{R}^2 , \quad f(x) = \begin{pmatrix} \cos x_2 \\ \sin x_1 \end{pmatrix}$

Clearly $D(f) = \mathbb{R}^2$ but since $-1 \leq \sin z , \cos z \leq 1$,



$R(f) = [-1, 1]^2$

Definition $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation if $\exists A \in \mathbb{R}^{m\times n}$ such that

$$T(x) = Ax$$

Note that not all mappings from $\mathbb{R}^n \to \mathbb{R}^m$ are matrix transformations. For instance $f(x) = x^2$, $x \in \mathbb{R}$ is not. Were it there would bee an $a \in \mathbb{R}$ s.t $f(x) = ax$.

EXAMPLE

$$T(x) \equiv \begin{pmatrix} 2x_1 - 7x_2 \\ 3x_1 + x_2 \\ x_2 \end{pmatrix}$$

Here $T : \mathbb{R}^2 \to \mathbb{R}^3$ and $T(x) = T_A(x) = Ax$ where

$$A = \begin{bmatrix} 2 & -7 \\ 3 & 1 \\ 0 & 1 \end{bmatrix}$$

If $x = (3,1)^T$ then its easy to compute $T(x) = \begin{pmatrix} -1 \\ 10 \\ 1 \end{pmatrix}$

Notation convention

By convention $T_A(x) \equiv Ax$. The subscript A is a reminder that T_A is a matrix transformation defined by $A \in \mathbb{R}^{m\times n}$.

EXAMPLE $T : \mathbb{R}^2 \to \mathbb{R}^2$ matrix transformation

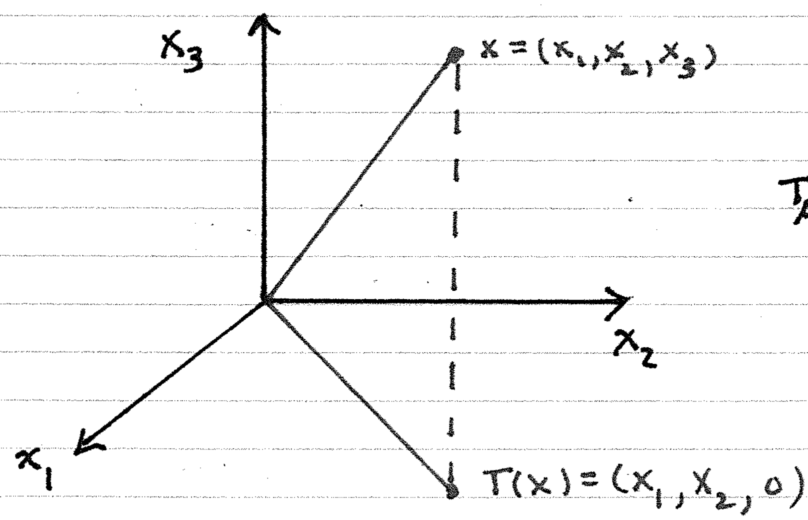
$$T_A(x) \equiv \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + x_2 \end{bmatrix}$$

Clearly the domain $D(T_A) = \mathbb{R}^2$. Noting

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$y = T(x)$ has a soln for all y $\Rightarrow$ range $R(T) = \mathbb{R}^2$

EXAMPLE Projection onto (x_1, x_2)-plane



$$T_A(x) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x$$

$D(T_A) = \mathbb{R}^3$

$R(T_A) = \{ x \in \mathbb{R}^3 : x_3 = 0 \}$ $\quad$ $x_3 = 0$ plane.

Language: $T_A(x)$ is the "image" of x under T_A

EXAMPLE Rotation in (x_1, x_2)-plane

$$T_\theta(x) \equiv Ax = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} x$$

To see why this is rotation of x we first express x in polar coordinates

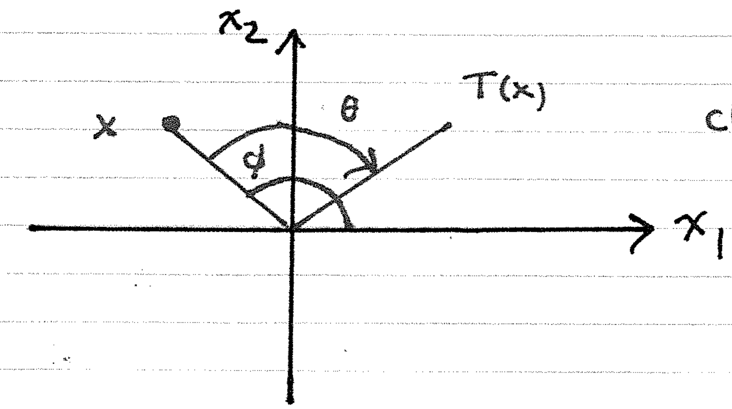
$$x = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$$

Expanding $T_\theta(x)$ yields

$$T_\theta(x) = \begin{pmatrix} r\cos\theta\cos\phi + r\sin\theta\sin\phi \\ r\cos\theta\sin\phi - r\sin\theta\cos\phi \end{pmatrix}$$

Using trig identities to simplify

$$T_\theta(x) = \begin{pmatrix} r\cos(\phi-\theta) \\ r\sin(\phi-\theta) \end{pmatrix}$$



clockwise.

Note how T_θ composes: $T_{\theta_1}(T_{\theta_2}(x)) = T_{\theta_1+\theta_2}(x)$

EXAMPLE Let $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is not known. But suppose we know the image of y, z under T_A. Specifically

$$(1) \qquad T_A(y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = u$$

$$(2) \qquad T_A(z) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = u$$

Collectively (1)–(2) are four eqns for (a, b, c, d):

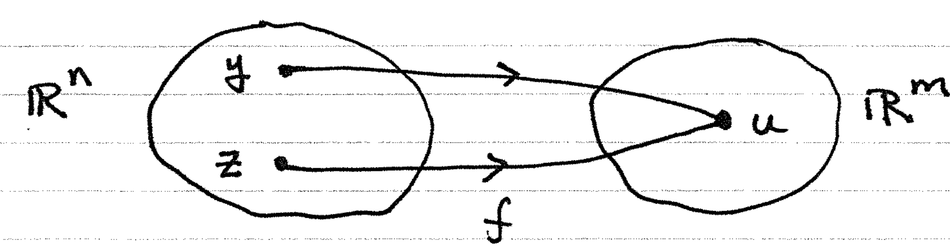
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solving we find $a = 0, b = 1, c = 0, d = 1$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

A final remark is A^{-1} DNE. Can be seen from

$$T_A(y) = T_A(z) \qquad y \neq z$$



Theorem $T: \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation if and only if the following hold for all $u, v \in \mathbb{R}^n$ and $k \in \mathbb{R}$

(1) $T(u+v) = T(u) + T(v)$ additive prop.

(2) $T(ku) = k\,T(u)$ homogeneity prop.

Pf First assume $T(x) = Ax$ for some $A \in \mathbb{R}^{m\times n}$. Then

$$T(u+v) = A(u+v) = Au + Av = T(u) + T(v).$$

$$T(ku) = A(ku) = kAu = k\,T(u)$$

Now assume T satisfies (1)-(2). We must show $\exists A$ such that

$$T(x) = Ax \qquad \forall x \in \mathbb{R}^n$$

Claim that (e_i are standard basis vectors)

(3) $$A = [\,T(e_1) \mid T(e_2) \cdots T(e_n)\,]$$

Noting $x = (x_1, \ldots x_n)^T$

$$\begin{aligned} Ax &= x_1 T(e_1) + x_2 T(e_2) + \cdots x_n T(e_n) && \text{property (2)} \\ &= T(x_1 e_1) + T(x_2 e_2) + \cdots T(x_n e_n) && \text{property (1)} \\ &= T(x_1 e_1 + \cdots x_n e_n) \\ &= T(x) \end{aligned}$$

$\square$

Remark Any transformation satisfying (1)-(2) in the theorem is said to be a linear transformation.

EXAMPLE A matrix transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$T_A\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} \qquad T_A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix}$$

Use these to compute $T_A(e_i)$, $i = 1, 2$.

$$T_A\begin{pmatrix}2\\1\end{pmatrix} - T_A\begin{pmatrix}1\\1\end{pmatrix} = T_A\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{e_1} = \begin{pmatrix}3\\1\end{pmatrix} - \begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}-2\\-1\end{pmatrix}$$

Similarly

$$T_A\begin{pmatrix}2\\1\end{pmatrix} - 2T_A\begin{pmatrix}1\\1\end{pmatrix} = -T_A\underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{e_2} = \begin{pmatrix}3\\1\end{pmatrix} - 2\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}-7\\-3\end{pmatrix}$$

Hence

$$A = [\, T(e_1) \mid T(e_2) \,]$$

$$A = \begin{bmatrix}-2 & 7\\ -1 & 3\end{bmatrix}$$

and $T_A(x) = Ax$.

Determinants

There are many ways to define the determinant det (A) of a square matrix A. Not found in our text but in "Matrices and Linear Transformations, C. Cullen 1966 is

Defn $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is defined by

(i) $\det(AB) = \det(A)\det(B) \qquad \forall A, B \in \mathbb{R}^{n\times n}$

(ii) $\det(E_1(k)) = k \qquad \forall k \in \mathbb{R}$

where $E_1(k)$ is the elementary matrix obtained by multiplying the 1rst row of I by $k \neq 0$.

Not done here but (i)-(ii) can be used to show how to compute matrices

EXAMPLE

$$\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = (2)(1) - (-1)(3) = 5$$

EXAMPLE (Cofactor Expansion)

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 2 & 2 \end{vmatrix} \end{array} = 1 \cdot \begin{vmatrix} -1 & 2 \\ 2 & 2 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} + 3 \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}$$

$$= (-6) - 2(-2) + 3(4)$$

$$= 10$$

Determinant Properties (without proof) $A \in \mathbb{R}^{n \times n}$

(1) $\det I = 1$

(2) $\det A^T = \det A$

(3) $\det(A^{-1}) = (\det A)^{-1}$

(4) $\det(A) = 0 \Rightarrow A^{-1}$ D.N.E.

(5) $\det(AB) = \det A \det B$

(6) $\det(cA) = c^n \det A \qquad c \in \mathbb{R}$

(7) A upper or lower triangular then

$$\det(A) = a_{11} a_{12} \cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$$

Note that $\det(A+B) \neq \det A + \det B$ as is illustrated in the following.

EXAMPLE

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad A+B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

Here

$$\det A + \det B = 1 + 8 = 9$$

$$\det(A+B) = 23$$

not equal