Vectors in $\mathbb{R}^n$ - Review

Notationally we write

$$x = (x_1, x_2, \ldots x_n) \qquad \in \mathbb{R}^n$$

where each x_k is a real number.
Standard basis vectors are

$$e_i = (0, 0, \ldots 1, 0, \ldots 0)$$
$$\Uparrow$$
$$i^{th} \text{ spot}$$

Thus x may be written as the sum

$$x = x_i e_i = x_1 e_1 + \cdots x_n e_n$$

Vectors in $\mathbb{R}^n$ satisfy the algebraic axioms of any vector space (see posted axioms)

Recall the following

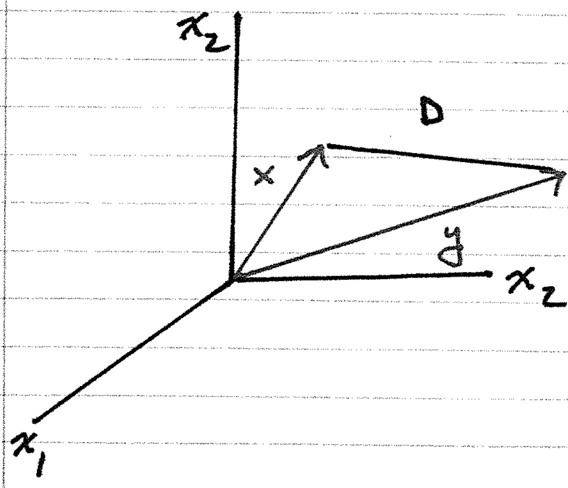
(1) $\quad x \cdot y = x_i y_i = x_1 y_1 + \ldots x_n y_n \qquad$ dot

(2) $\quad \|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + \cdots x_n^2} \qquad$ norm

(3) $\quad d(x,y) = \|x - y\| \qquad$ distance

Sometimes (1) is referred to as the <u>Euclidean dot product</u>

EXAMPLE Compute distance between
$x = (1, 1, 1)$ and $y = (3, 2, 4)$



$$y - x = (2, 1, 3)$$

$$D = \|y - x\|$$

$$D = \sqrt{2^2 + 1^2 + 3^2}$$

$$D = \sqrt{14}$$

Some important Results

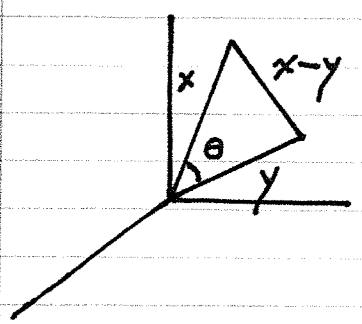
(1) $\quad x \cdot y = \|x\| \, \|y\| \cos\theta$

(2) $\quad |x \cdot y| \leq \|x\| \, \|y\|$ Cauchy Schwartz

(3) $\quad \|x + y\| \leq \|x\| + \|y\|$ Triangle Inequality

(4) $\quad d(x, y) \leq d(x, z) + d(z, y)$

Pf of (1)



Triangle identity

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\,\|y\|\cos\theta$$

expand and cancel with terms here

$$-2\, x \cdot y = -2\,\|x\|\,\|y\|\cos\theta. \quad \square$$

Pf of (2) Follows immediately from (1)

$$|x \cdot y| = \|x\| \, \|y\| \, |\cos\theta| \leq \|x\| \, \|y\|$$

since $|\cos\theta| \leq 1$.

Pf of (3) Follows from Cauchy Schwartz inequality

$$\|x+y\|^2 = (x+y)\cdot(x+y)$$

$$= \|x\|^2 + 2\,x\cdot y + \|y\|^2$$

$$\leq \|x\|^2 + 2|x\cdot y| + \|y\|^2$$

C.S.

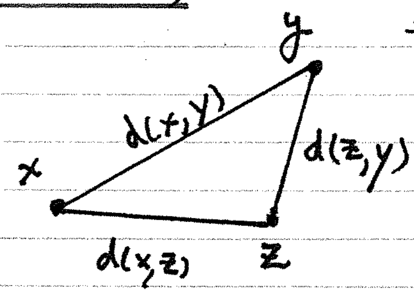
$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

Take square roots yields

$$\|x+y\| \leq \|x\| + \|y\|$$

Pf of (4) Our notion of distance suggest (4) to be true - reference diagram



Noting $d(x,y) = \|x-y\|$

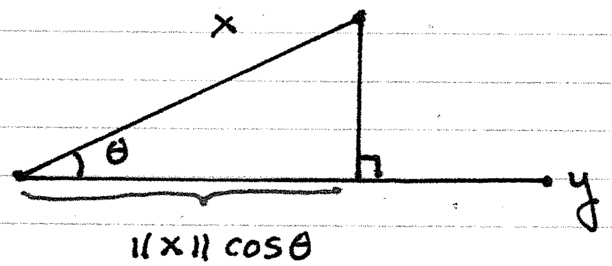
$$\|x-y\| = \|\underline{x-z} + \underline{z-y}\|$$

$$\leq \|x-z\| + \|z-y\|$$

$$\leq d(x,z) + d(z,y)$$

Projections in $\mathbb{R}^3$

We seek to find the projection of x onto y.



$$\cos\theta = \frac{x \cdot y}{\|x\| \, \|y\|}$$

Letting $\hat{y}$ be the unit vector in direction of y is:

$$\hat{y} = \frac{y}{\|y\|}$$

Then the projection of x onto y

$$\text{proj}_y x = \|a\| \cos\theta \, \hat{y}$$

$$= \frac{\cancel{\|x\|} \cdot (x \cdot y)}{\|y\|^2 \, \cancel{\|x\|}} \, y$$

yields (after cancellation)

$$\boxed{\text{proj}_y x = \left(\frac{x \cdot y}{y \cdot y}\right) y}$$

Projection as a linear transformation

Seek a (matrix) transformation $T_u : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T_u(x) = \text{projection of } x \text{ onto } u$$

Theorem Let $u \in \mathbb{R}^3$ be a unit vector. Then

$$T_u(x) = Ax = \text{proj}_u x \quad , \quad A \equiv uu^T$$

Pf : Direct calculations. Given $A = uu^T$, $\|u\| = 1$

$$Ax = \begin{bmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2u_2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3u_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} (u\cdot x)\, u_1 \\ (u\cdot x)\, u_2 \\ (u\cdot x)\, u_3 \end{bmatrix}$$

$$= (u\cdot x)\, u \qquad \square$$

Remark: Can easily show $T_u(T_u(x)) = T_u(x)$

$$A^2x = A(Ax) = A\left((u\cdot x)\, u\right)$$

$$= (u\cdot x)\, Au$$

$$= (u\cdot x)\, \overset{1}{\cancel{(u\cdot u)}}\, u = (u\cdot x)\, u = Ax$$