Vector Space Definition Let V be a nonempty set on which the operations of addition $+$ and scalar multiplication have been defined:

(i) $\mathbf{u}+\mathbf{v}$ is defined $\forall \mathbf{u}, \mathbf{v} \in V$

(ii) $c\mathbf{u}$ is defined $\forall \mathbf{u} \in V, \forall c \in \mathbb{R}$.

The set V is called a vector space if additionally, $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\forall b, c \in \mathbb{R}$ the following axioms hold:

(A1)	$\mathbf{u}+\mathbf{v} \in V$	V closed under addition
(A2)	$\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$	addition is commutative
(A3)	$\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$	addition is associative
(A4)	$\exists \mathbf{0} \in V \textit{such that } \mathbf{u}+\mathbf{0} = \mathbf{u}$	existence of a zero vector
(A5)	$\exists -\mathbf{u} \in V \textit{ such that } \mathbf{u}+(-\mathbf{u}) = \mathbf{0}$	existence of a negative element
(A6)	$c\mathbf{u} \in V$	closed under scalar multiplication
(A7)	$c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$	distributive property I
(A8)	$(b+c)\mathbf{u} = b\mathbf{u}+c\mathbf{u}$	distributive property II
(A9)	$c(\beta\mathbf{u}) = (c\beta)\mathbf{u}$	commutativity of scalar multiplication
(A10)	$1\mathbf{u} = \mathbf{u}$	scalar multiplication identity element

Sometimes the symbols $\oplus$ and $\odot$ will be used to denote vector addition and scalar multiplication, respectively. This will only be done when vector addition and scalar multiplication is defined in some (confusing) nontraditional way. Then $\mathbf{u}+\mathbf{v}$ will be written $\mathbf{u} \oplus \mathbf{v}$ and $c\mathbf{u}$ will be written $c \odot \mathbf{u}$. As one example

$$3(\mathbf{u}+\mathbf{v}) = 3 \odot (\mathbf{u} \oplus \mathbf{v})$$

EXAMPLE $V = \mathbb{R}^n$ is a vector space if

$$u + v \equiv (u_1 + v_1, \ldots\ldots u_n + v_n)$$

$$ku \equiv (ku_{1,}, \ldots\ k u_n) \qquad \forall k \in \mathbb{R}$$

where $\vec{0} = (0, \ldots .0)$. Note V is closed under addition (A1) and scalar multiplication (A6).

EXAMPLE Spaces of real matrices (for $n=2$)

$$V = M_{22} = \text{set of all } 2\times 2 \text{ matrices } \mathbb{R}^{2\times 2}$$

$$u + v \equiv \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix} \quad \text{matrix addition}$$

$$ku \equiv \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

where in this case

$$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

With these definitions M_{22} is a vector space

Clearly

$$M_{mn} = \text{set of all } A \in \mathbb{R}^{m\times n}$$

is as well.

EXAMPLE $\quad V = P_n$

$P_n \equiv$ set of all n^{th} degree polynomials

Here $u \in P_n \Rightarrow \exists\, a_k$ such that

$$u(x) = a_n x^n + \cdots a_1 x + a_0 \qquad x \in \mathbb{R}$$

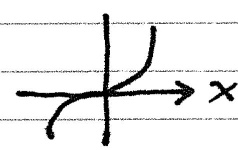
Usual addition and scalar multiplication $\Rightarrow P_n$ is a vector space.

EXAMPLE $\quad V = C^n[a,b]$ or $C^n(I) \quad I = [a,b]$

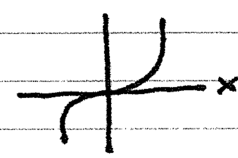
Is the set of $f(x)$ defined on $[a,b]$ that are n times continuously differentiable.
When $n=0$, $C^0[a,b] \equiv C[a,b]$ the space of continuous functions on $[a,b]$.

With regular addition and mult. all C^n are vector spaces.

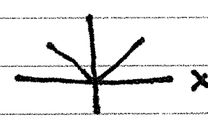
$$f(x) = \begin{cases} \frac{1}{6}x^3 & x>0 \\ -\frac{1}{6}x^3 & x<0 \end{cases}$$



$$f'(x) = \begin{cases} \frac{1}{2}x^2 & x>0 \\ -\frac{1}{2}x^2 & x<0 \end{cases}$$



$$f''(x) = |x|$$



For this $f(x)$, $f \in C^2(\mathbb{R})$, $f' \in C^1(\mathbb{R})$, $f'' \in C(\mathbb{R})$

Note the nesting of these vector spaces

$$C^n(I) \cdots \subset C^2(I) \subset C^1(I) \subset C(I)$$

EXAMPLE Zero Vector

$$V = \{x : x \in \mathbb{R}^n, x = 0\} = \{0\} \quad \text{(one element)}$$

EXAMPLE Line thru origin

$$V = \{(x_1, x_2, x_3) : \exists t \in \mathbb{R}, x_k = a_k t\}$$

EXAMPLE Nullspace $V = N(A)$ $\qquad A \in \mathbb{R}^{n \times n}$

$$N(A) = \{x \in \mathbb{R}^n : Ax = 0\}$$

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Let $u, v \in N(A)$. Then $Au = 0$ and $Av = 0$.

$$Au + Av = A(u+v) = 0 \quad \Rightarrow u+v \in N(A)$$

$$A(ku) = kAu = 0 \quad \Rightarrow ku \in N(A), \forall k \in \mathbb{R}$$

EXAMPLE Planes in $\mathbb{R}^3$

$$V_1 = \{x \in \mathbb{R}^3 : ax_1 + bx_2 + cx_3 = 0\}$$

$$V_2 = \{x \in \mathbb{R}^3 : ax_1 + bx_2 + cx_3 = 1\}$$

Here V_1 is a vectorspace but V_2 is not.

V_2 not closed w.r.t. addition. Let $u, v \in V_2$ and $w \equiv u+v$

$$\begin{array}{rl} au_1 + bu_2 + cu_3 &= 1 \\ av_1 + bv_2 + cv_3 &= 1 \\ \hline aw_1 + bw_2 + cw_3 &= 2 \end{array} \quad \text{add}$$

$$\Rightarrow w \notin V_2$$

EXAMPLE $V = \{u \in \mathbb{R}^{2\times 2} : u^T = u\}$

Set of symmetric real 2×2 matrices.

EXAMPLE $V = \{f : f(x) = a_1 + a_2 \sin x \text{ for some } a_k\}$

First two terms of a Fourier series.

EXAMPLE $V = \mathbb{R}^{n\times n}$ with addition redefined

Let $u, v \in V$ and $k \in \mathbb{R}$.

$$u \oplus v \equiv uv \qquad \text{matrix product}$$

$$k \odot v \equiv ku$$

$$\vec{0} \equiv I \qquad \text{identity}$$

Addition is closed and associative but

* $$u \oplus v \neq v \oplus u \qquad \forall u, v.$$

Existence of $\vec{0}$ is satisfied

$$u \oplus \vec{0} = uI = u$$

There is not always $-u \in V$ s.t. $u + (-u) = \vec{0}$.
Were there such a $v \equiv -u$ then

* $$u \oplus v = uv = I \Rightarrow v = u^{-1} \text{ inverse}$$

but not all matrices have an inverse.

Some scalar properties (A6)-(A9) are not satisfied either.

EXAMPLE Unusual vector space

$$\mathbb{V} \equiv \mathbb{R}^+ = \text{set of } \underline{\text{nonnegative}} \text{ real numbers}$$

$$\vec{u}+\vec{v} \equiv uv \qquad u,v \geq 0$$

$$k\vec{u} \equiv u^k \qquad u \geq 0 \ , \ k \in \mathbb{R}$$

$$\vec{0} \equiv 1$$

First note that $\mathbb{V}$ is closed under vector addition and scalar multiplication since $u+v \geq 0$ and $ku \geq 0$. Thus (A1) and (A6) are satisfied.

(A2) $\quad \vec{u}+\vec{v} = \vec{v}+\vec{u} = uv$

Addition is also associative

(A3) $\quad \vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w} = uvw$

The existence of a zero vector and negative element is more interesting. Again note $\vec{0} = 1$ and $-\vec{u} = u^{-1}$

(A4) $\quad \vec{u}+\vec{0} = 1 \cdot u = u = \vec{u}$

(A5) $\quad \vec{u}+(-\vec{u}) = u \cdot \frac{1}{u} = 1 = \vec{0}$

Finally

(A7) $\quad c(\vec{u}+\vec{v}) = (uv)^c = u^c v^c = c\vec{u}+c\vec{v}$

(A8) $\quad (b+c)\vec{u} = u^{(b+c)} = u^b u^c = b\vec{u}+c\vec{u}$

(A9) $\quad c(b\vec{u}) = (u^b)^c = u^{bc} = (bc)\vec{u}$

(A10) $\quad 1\vec{u} = u^1 = u = \vec{u} \qquad \square$

EXAMPLE Solution spaces of linear ODE's

We consider solutions to a second order linear homogeneous differential equation.

$$V = \{ u \in C^2[0,1] : u'' + p(x)u' + q(x)u = 0 \}$$

with the usual addition and scalar multiplication.

From the general theory of ODE's we know V must be a sum of two indepent solutions, i.e. if $u \in V$ then $\exists c_K$ such that

$$u(x) = c_1 u_1(x) + c_2 u_2(x)$$

EXAMPLE (Not a vector space) $V = \mathbb{R}$

Reverse the usual operations of addition and multiplication on $V = \mathbb{R}^+$

$$\vec{u} \oplus \vec{v} \equiv uv$$

$$k \odot \vec{u} \equiv k + v$$

$$\vec{0} \equiv 1$$

$$-\vec{u} \equiv \frac{1}{u} \qquad u \neq 0$$

One can verify this is <u>not</u> a vector space. (A8) ??

$$(b+c)\,\vec{u} = (b+c+u)^*$$

$$\left.\begin{aligned} b\,\vec{u} &= b + u \\ c\,\vec{u} &= c + u \end{aligned}\right\} \text{sum does not equal } *$$

EXTRA ODD (POTENTIAL) SPACES

① $V \equiv \mathbb{R}^2$

$$\vec{u} + \vec{v} \equiv (u_1 v_1 , u_2 v_2)$$

$$k\vec{u} \equiv (u_1 , k u_2)$$

$$\vec{0} \equiv (1, 1)$$

② $V \equiv \mathbb{R}^2$

Different addition/scalar multiplication

$$\vec{u} + \vec{v} \equiv (u_1 + u_2 , v_1 + v_2) \qquad k\vec{u} \equiv (2ku_1 , 2ku_2)$$

$$\vec{u} + \vec{v} \equiv (u_1 + v_1 , u_2 + v_2) \qquad k\vec{u} \equiv (ku_1 , ku_2)$$

$$\vec{u} + \vec{v} \equiv (u_1 + u_2 , v_1 + v_2) \qquad k\vec{u} = \vec{0}$$

Theorem Cancellation Law for Vector Addition
Let V be a vector space and $x, y, z \in V$

$$x + z = y + z \Rightarrow x = y$$

Pf/

$$\begin{aligned} x &= x + 0 \\ &= x + (z + (-z)) \\ &= (x + z) + (-z) \\ &= (y + z) + (-z) \\ &= y + (z + (-z)) \\ &= y + 0 \\ &= y \end{aligned}$$

$\square$

Theorem Let $a, b, c \in V$. Then

$$a = b \Rightarrow a + c = b + c$$

Pf/

$$a = b$$

$$a + 0 = b + 0$$

$$(a + c) + (-c) = (b + c) + (-c)$$

$$x + z = y + z$$ In thm above!

Hence $x = y$ or $a + c = b + c$ $\square$

Theorem Let V be a vector space.
Let $\vec{u} \in V$ and $k \in \mathbb{R}$. Then

(i) $0\vec{u} = \vec{0}$

(ii) $k\vec{0} = \vec{0}$

(iii) $(-1)\vec{u} = -\vec{u}$

(iv) $k\vec{u} = \vec{0} \Rightarrow k = 0$ or $\vec{u} = \vec{0}$

Proof of (i) only

Closure properties (A1)+(A6) $\Rightarrow 0\vec{u} + 0\vec{u} \in V$

$0\vec{u} + 0\vec{u} = (0+0)\vec{u}$ (A8)

$0\vec{u} + 0\vec{u} = 0\vec{u}$ prop. of real nos.

By (A5) there exists an $-0\vec{u} \in V$

$-0\vec{u} + (0\vec{u} + 0\vec{u}) = 0\vec{u} + (-0\vec{u})$ (A2), (A5)

$(-0\vec{u} + 0\vec{u}) + 0\vec{u} = \vec{0}$ (A5), (A3)

$\vec{0} + 0\vec{u} = \vec{0}$ (A5)

$0\vec{u} = \vec{0}$ (A4)

In all we used (A1) - (A6) and (A7) to prove this annoyingly obvious fact :)

EXAMPLE Careful vector algebra

Let V be *any* vector space. Solve for u

$$2u + v = 3w$$

requires justification (later in notes)

$$(2u + v) + (-v) = 3w + (-v)$$

(A3)

$$2u + (v + (-v)) = 3w + (-v)$$

(A5)

$$2u + 0 = 3w + (-v)$$

(A4)

$$2u = 3w + (-v)$$

$$\tfrac{1}{2}(2u) = \tfrac{1}{2}(3w + (-v))$$

(A9)

$$(\tfrac{1}{2} \cdot 2)u = \tfrac{1}{2}(3w + (-v))$$

real numbers

$$1 \cdot u = \tfrac{1}{2}(3w + (-v))$$

(A10)

$$u = \tfrac{1}{2}(3w + (-v))$$

□

So, in all we used axioms

(A3) (A4) (A5) (A9) (A10)