**Definition**: Let $V$ be a vector space. 
$W \subseteq V$ is a subspace of $V$ if $W$ is also a vector space using same $+$ and $\cdot$ operations as $V$.

Above, $W \subseteq V$ means $W$ is a subset of $V$.

Just being a subset of $V$ is not enough to assure $W$ is a subspace.

**Theorem**: $W \subseteq V$ is a subspace of $V$ if and only if

(i) $u, v \in W \Rightarrow u + v \in W$

(ii) $u \in W, k \in \mathbb{R} \Rightarrow ku \in W$

**Proof**: Text

**Remarks**

Hence to show $W$ is a subspace it suffices to show it is closed with respect to $+$ and $\cdot$.

One need not check the other axioms that define a vector space.
EXAMPLE  Geometry in \( \mathbb{R}^3 \)

\( W = \{ \vec{0} \} \), lines and planes thru origin \( \vec{0} \)
are all subspaces of \( \mathbb{R}^3 \)

EXAMPLE  \( W = \mathbb{Z}^2 \)  \( V = \mathbb{R}^2 \)

Recall \( \mathbb{Z} \) = set of integers. Any element \( w \in W \) is an ordered pair of integers
\[ w = (m, n) \quad m, n \in \mathbb{Z} \]

Interestingly enough \( W \) is closed under +
It is not under scalar multiplication. Let \( \vec{u} \in W \)
\[ k \vec{u} \notin W \quad \text{for all } k \]
Just take \( k = 1 \), \( \vec{u} = (1, 3) \) to see this.
**Example:** Polynomials

\[ P_n = \{ p(x) : p(x) = a_0 + a_1 x + \cdots + a_n x^n \} \]

\( P_m \) a subspace of \( P_n \) if \( m \leq n \)

**Example:** Function spaces

Recall \( C^n(\mathbb{R}) \) is the set of all \( n \)-times continuously differentiable functions on \( \mathbb{R} \). Also recall

\[ f'(x) \text{ continuous } \implies f(x) \text{ continuous} \]

\[ f''(x) \text{ continuous } \implies f'(x) \text{ continuous} \]

Thus we have the following nesting of subspaces

\[ P_n \subset C^n(\mathbb{R}) \subset \cdots \subset C^1(\mathbb{R}) \subset C(\mathbb{R}) \]

**Example:** Nullspace \( W = N(A) = \{ x : Ax = 0 \} \)

Note \( N(A) \) nonempty since \( 0 \in N(A) \).

Let \( u, v \in N(A) \)

\[ A(u + v) = A(u + v) = 0 \implies u + v \in N(A) \]

Closure under scalar multiplication.

\[ kAu = A(ku) = 0 \implies ku \in N(A) \]
**Example**

Range of a matrix transformation

\[ T(x) = Ax \quad A \in \mathbb{R}^{m \times n}, \quad x \in \mathbb{R}^n \]

The range \( R(T) \) of \( T \) is defined as:

\[ R(T) = \{ y : \ y = T(x) \ \text{for some} \ x \} \]

We show \( R(T) \) is a subspace of \( \mathbb{R}^n \)

Let \( y_1, y_2 \in R(T) \). Then there are \( x_1, x_2 \) such that

\[ y_1 = T(x_1) \]
\[ y_2 = T(x_2) \]
\[ y_1 + y_2 = T(x_1) + T(x_2) = T(x_1 + x_2) \]

1. Shows \( R(T) \) is closed under addition

\[ y = T(x) \quad \text{property of} \ T \]
\[ ky = kT(x) = T(kx) \]

2. Shows \( R(T) \) is closed under scalar multiplication.

Collectively, 1)–2) \( \Rightarrow \) \( R(T) \) subspace.
EXAMPLE Functions with zero average

\[ W = \{ u \in C[a, b] : \int_a^b u(x) \, dx = 0 \} \]

Let \( u_1, u_2 \in W \implies \)

\[ \int_a^b u_1(x) + u_2(x) \, dx = \int_a^b u_1(x) \, dx + \int_a^b u_2(x) \, dx = 0 \]

implies closure under +. Clearly closed under \( \circ \).

EXAMPLE

\[ W = \{ u \in M_{22} : u \text{ upper triangular} \} \]

Clearly sums of triangular matrices are triangular.

\( W \) subspace of \( M_{22} \).
**Definition**  
A vector \( \mathbf{v} \in \mathcal{V} \) is a linear combination of \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) if there exist scalars \( c_1, \ldots, c_n \) such that

\[
\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n
\]

**Definition**  
Let \( S = \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \} \) be a nonempty set of vectors in vector space \( \mathcal{V} \). The set of all possible linear combinations of \( \mathbf{v}_k \) is denoted

\[
W = \text{span}(S)
\]

\[
W = \text{span}\{ \mathbf{v}_1, \ldots, \mathbf{v}_n \}
\]

**Theorem**  
\( W = \text{span}(S) \) is a subspace of \( \mathcal{V} \)

**Example**  
\( \mathcal{V} = \mathbb{R}^2 \); \( \mathbf{v}_1 = (1, 1) \), \( \mathbf{v}_2 = (0, 7) \), \( \mathbf{v}_3 = (3, 1) \)

\[
W = \text{span}\{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} = \mathbb{R}^2
\]

**Example**  
\( \mathcal{V} = P_2 \); \( \mathbf{v}_1 = x^2 - x \), \( \mathbf{v}_2 = x \)

\[
W = \text{span}\{ \mathbf{v}_1, \mathbf{v}_2 \}
\]

Clearly \( \mathbf{v}_3 = 1 \notin W \) so \( W \subset \mathcal{V} \) proper subset and

\[
W = \{ \mathbf{v} \in P_2 : \mathbf{v}(0) = 0 \}
\]
**Example** \( V = M_{22} \) set of all \( 2 \times 2 \) real matrices

\[
W = \{ v \in M_{22} : v = \begin{bmatrix} a & b \\ a+b & b \end{bmatrix} \text{ for some } a, b \in \mathbb{R} \} \]

Note that any \( v \in W \) can be written

\[
W = a \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = a v_1 + b v_2
\]

thus \( W = \text{span}\{v_1, v_2\} \) as well.

**Example** Show \( v = (1, 3, -3) \) is a linear combination of \( v_1 = (1, 2, -1) \) and \( v_2 = (3, 1, 1) \)

\[
v = c_1 v_1 + c_2 v_2
\]

yields a linear system for \( c = (c_1, c_2) \)

\[
c_1 = 2 \quad c_2 = -1
\]

**Example** \( V = P_2 \). Express \( u(x) = 6x^2 + 12x + 6 \) as a linear combination of \( v_1(x) = 2x^2 + x + 2 \), \( v_2(x) = x^2 - x + 1 \), if possible

\[
u = c_1 v_1 + c_2 v_2 \quad \text{(equate powers of } x)\]

Leads to

\[
\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 6 \end{bmatrix}
\]

Consistent system: \( c_1 = 6, \ c_2 = -6 \)
EXAMPLE  Let $\mathcal{V} = \mathbb{R}^3$ and $W = \text{span}\{v_1, v_2, v_3\}$ where

$$v_1 = (1, 1, 2), \quad v_2 = (1, 0, 1), \quad v_3 = (2, 1, 3)$$

Does $W = \mathcal{V}$?

The question is, can every $b \in \mathcal{V}$ be written as a linear combination of $v_k$:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = b$$

As a system of equations (in matrix form)

$$\begin{bmatrix}
1 & 1 & 2 \\
1 & 0 & 1 \\
2 & 1 & 3
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
= 
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
$$

The system $Ac = b$ has a solution for all $b \in \mathbb{R}^3$ only if $A$ is invertible. But

$$\det A = 0$$

Since $\text{N}(A) \neq \{0\}$, the column space is not $\mathbb{R}^3$. One can show

$$A \sim \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}$$

Columns 1+2 of original matrix for basis

$$\text{col}(A) = \text{span}\{v_1, v_2\}$$
**Linear Independence**

**Definition:** \( S = \{ v_1, v_2, \ldots, v_r \} \subset V \) is linearly independent if

\[
c_1 v_1 + c_2 v_2 + \cdots + c_r v_r = 0
\]

implies \( c_k = 0 \) for all \( k = 1, 2, \ldots, r \).

**Remark**  
\( \text{span}(S) \) is always a subspace of \( V \) but many vectors in \( S \) may be redundant in the sense that if you remove one (or two) the "size" of \( \text{span}(S') \) is unchanged.  

Linear independence is a concept meant to measure this "redundancy". If \( S \) is linearly independent then it is in some sense the "smallest" set needed to describe \( W = \text{span}(S) \).

**Example**  
\( V = \mathbb{R}^2 \quad v_1 = (1, 2) \quad v_2 = (3, 6) \)

\[
S = \{ v_1, v_2 \} \quad \text{dependent}
\]

\[
S' = \{ v_1 \} \quad \text{independent}
\]

Here dependence is easily seen since \( v_2 = 3v_1 \), or

\[
c_1 v_1 + c_2 v_2 = 0
\]

if \( c_1 = 3 \quad c_2 = -1 \) (non zero!)

Also note \( \text{span}(S) = \text{span}(S') \).
EXAMPLE  Show the following set is linearly independent

\[ S = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \} \quad \vec{v}_1 = (1, 1, 0) \quad \vec{v}_2 = (1, 0, -1) \quad \vec{v}_3 = (1, 1, 1) \]

Recall, \( S \) is independent if

\[ c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0} \quad \Rightarrow c_k = 0 \]

This is equivalent to the linear system

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[ \vec{c} = \vec{0} \quad \text{only if } A \text{ is invertible.} \]

\[ \det A = -1 \]

hence, \( S \) is independent.

EXAMPLE  \( V = \mathbb{R}^n, \ S = \{ \vec{v}_1, \ldots, \vec{v}_r \} \) generalization of previous example

\[ c_1 \vec{v}_1 + \cdots + c_r \vec{v}_r = \vec{0} \]

\[ A \vec{c} = \begin{bmatrix}
\vec{v}_1 \\
\vec{v}_2 \\
\vdots \\
\vec{v}_r
\end{bmatrix} \vec{c} = \vec{0} \quad A \in \mathbb{R}^{n \times r} \]

Hence, \( S \) independent \( \Leftrightarrow \) \( \text{null}(A) = \{ \vec{0} \} \)
EXAMPLE \( V = P_2 \), \( \beta = \{ p_1, p_2, p_3 \} \) where

\[
    \begin{align*}
    p_1 &= 2x \\
    p_2 &= x^2 + 1 \\
    p_3 &= 2x^2 - 2x + 2
    \end{align*}
\]

Linear independence

(1) \( c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) = 0 \) \( \forall x \)

implies \( c_1 = 0 \). Using \( p_k \) definitions (1)

\[
    (c_2 + 2c_3)x^2 + (2c_1 - 2c_3)x + (c_2 + 2c_3) = 0
\]

Coefficients must vanish

\[
    A \mathbf{c} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{0}
\]

Since \( \det(A) = 0 \) (repeated rows), \( N(A) \neq \{ \mathbf{0} \} \)

so the set is dependent

Can show \( A \sim \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix} \) so that \( p_3 \) is a linear comb of \( p_1, p_2 \):

\[
    p_3 = 2p_2 - p_1
\]

Thus

\[
    \text{span}\{p_1, p_2, p_3\} = \text{span}\{p_1, p_2\}
\]
Linear Independence of Functions

**Theorem** Let \( V = C^n(I) \) and \( S = \{u_1, \ldots, u_n\} \subseteq V \). \( S \) is independent on \( I \) if the Wronskian

\[
W(x) = \begin{vmatrix}
u_1 & u_2 & \ldots & u_n \\
u_1' & u_2' & \ldots & u_n' \\
\vdots & \vdots & \ddots & \vdots \\
u_1^{(n-1)} & u_2^{(n-1)} & \ldots & u_n^{(n-1)}
\end{vmatrix} \neq 0 \quad \forall x \in I
\]

**Pf:** Do in class.

**Example** \( u_1 = x^2 + x, \quad u_2 = x - 1, \quad u_3 = x^2 + 3x - 2 \)

\[
W(x) = \begin{vmatrix}
x^2 + x & x - 1 & x^2 + 3x - 2 \\
x + 1 & 1 & 2x + 3 \\
2 & 0 & 2
\end{vmatrix} = 0
\]

Conclude \( S = \{u_1, u_2, u_3\} \) is a dependent set.

**Example** For what \( \lambda \in \mathbb{R} \) are the following independent

\( u_1 = x^2 + x + 1, \quad u_2 = x - \lambda, \quad u_3 = 2x^2 - 2 + 1 \)

After some calculations

\[
W(x) = 6\lambda + 2 = 0 \iff \lambda = -\frac{1}{3}
\]

**Example** \( S = \{1, \sin x, \sin 2x\} \)

Using addition/multi trig identities

\[
W(x) = -3\sin x - \sin 3x
\]

Independent on "most" intervals.