Definition: Let V be a vector space. $W \subset V$ is a subspace of V if W is also a vector space using same $+$ and $\cdot$ operations as V

Above, $W \subset V$ means W is a subset of V.

Just being a subset of V is not enough to assure W is a subspace.

Theorem $W \subset V$ is a subspace of V if and only if

$$\text{(i)} \quad u, v \in W \Rightarrow u + v \in W$$

$$\text{(ii)} \quad u \in W, k \in \mathbb{R} \Rightarrow ku \in W$$

Proof: Text

Remarks

Hence to show W is a subspace it suffices to show it is closed with respect to $+$ and $\cdot$.

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One need not check the other axioms that define a vector space.

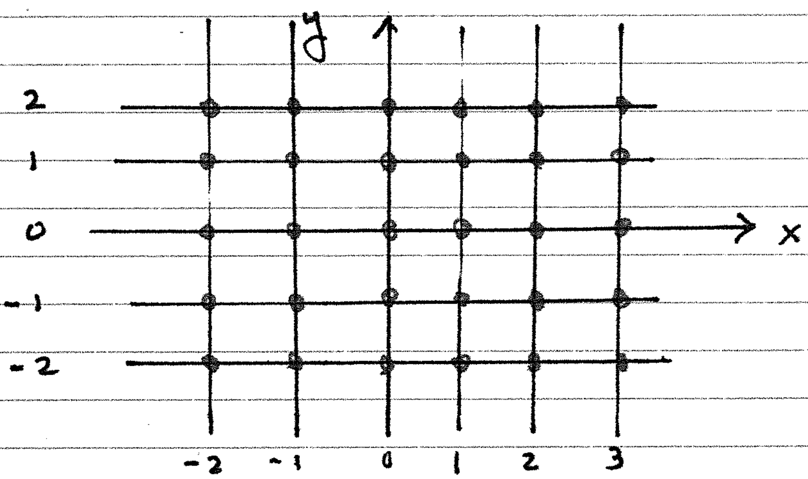
EXAMPLE Geometry in $\mathbb{R}^3$

$W = \{\vec{0}\}$, lines and planes thru origin $\vec{0}$ are all subspaces of $\mathbb{R}^3$

EXAMPLE $W = \mathbb{Z}^2$ $V = \mathbb{R}^2$

Recall $\mathbb{Z}$ = set of integers. Any element $w \in W$ is an ordered pair of integers

$$w = (m, n) \qquad m, n \in \mathbb{Z}$$



Dots are elements of W

Interestingly enough W <u>is</u> closed under +

It is not under scalar multiplication. Let $\vec{u} \in W$

$$k\vec{u} \notin W \qquad \text{forall } k$$

Just take $k=1$, $\vec{u} = (1,3)$ to see this.

EXAMPLE Polynomials

$$P_n \equiv \{ p(x) : p(x) = a_0 + a_1 x + \cdots a_n x^n \}$$

P_m a subspace of P_n if $m \leq n$

EXAMPLE Function spaces

Recall $C^n(I)$ is the set of all n-times continuously differentiable functions on I. Also recall

$$f'(x) \text{ continuous} \Rightarrow f(x) \text{ continuous}$$

$$f''(x) \text{ continuos} \Rightarrow f'(x) \text{ continuous}$$

Thus we have the following nesting of subspaces

$$P_n \subset C^\infty(\mathbb{R}) \subset \cdots C^n(\mathbb{R}) \subset \cdots C'(\mathbb{R}) \subset C(\mathbb{R})$$

EXAMPLE Nullspace $W = N(A) = \{ x : Ax = 0 \}$

Note $N(A)$ nonempty since $\vec{0} \in N(A)$.

Let $u, v \in N(A)$

$$Au + Av = A(u+v) = 0 \Rightarrow u+v \in N(A)$$

Closure under scalar multiplication.

$$kAu = A(ku) = 0 \Rightarrow ku \in N(A)$$

EXAMPLE Range of a matrix transformation

$$T(x) \equiv Ax \qquad A \in \mathbb{R}^{n \times n}, \quad x \in \mathbb{R}^n$$

The range $R(T)$ of T is defined as:

$$R(T) = \{ y : y = T(x) \text{ for some } x \}$$

We show $R(T)$ is a subspace of $\mathbb{R}^n$

Let $y_1, y_2 \in R(T)$. Then there are x_1, x_2 such that

$$y_1 = T(x_1)$$

$$y_2 = T(x_2)$$

$$y_1 + y_2 = T(x_1) + T(x_2) \overset{\text{property of } T}{=} T(x_1 + x_2)$$

1) Shows $R(T)$ is closed under addition

$$y = T(x)$$

$$ky = k\,T(x) \overset{\text{property of } T}{=} T(kx)$$

2) Shows $R(T)$ is closed under scalar multiplication.

Collectively, 1)–2) $\Rightarrow$ $R(T)$ subspace.

EXAMPLE Functions with zero average

$$W = \{ u \in C[a,b] : \int_a^b u(x)\,dx = 0 \}$$

Let $u_1, u_2 \in W \Rightarrow$

$$\int_a^b u_1(x) + u_2(x)\,dx = \int_a^b u_1(x)\,dx + \int_a^b u_2(x)\,dx = 0$$

implies closure under $+$. Clearly closed under $\odot$

EXAMPLE

$$W = \{ u \in M_{22} : u \text{ upper triangular} \}$$

Clearly sums of triangular matrices are triangular.

W subspace of M_{22}

DEFINITION A vector $v \in V$ is a <u>linear combination</u> of $v_1, v_2 \ldots v_n$ if there exist scalars $c_1, \ldots c_n$ such that

$$v = c_1 v_1 + c_2 v_2 + \cdots c_n v_n$$

DEFINITION Let $S = \{v_1, \ldots v_n\}$ be a nonempty set of vectors in vector space V. The set of all possible linear combinations of v_k is denoted

$$W = \text{span}(S)$$

$$W = \text{span}\{v_1, \ldots v_n\}$$

<u>Theorem</u> $W = \text{span}(S)$ is a subspace of V

EXAMPLE $V = \mathbb{R}^2$; $v_1 = (1,1)$, $v_2 = (0,7)$, $v_3 = (3,1)$

$$W = \text{span}\{v_1, v_2, v_3\} = \mathbb{R}^2$$

EXAMPLE $V = P_2$; $v_1 = x^2 - x$, $v_2 = x$

$$W \equiv \text{span}\{v_1, v_2\}$$

Clearly $v_3 = 1 \notin W$ so $W \subset V$ proper subset and

$$W = \{v \in P_2 : v(0) = 0\}$$

EXAMPLE $V = M_{22}$ set of all 2x2 real matrices

$$W \equiv \left\{ v \in M_{22} : v = \begin{bmatrix} a & b \\ a+b & b \end{bmatrix} \text{ for some } a, b \in \mathbb{R} \right\}$$

Note that any $v \in W$ can be written

$$W = a \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}_{v_1} + b \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{v_2} = a v_1 + b v_2$$

thus $W = \text{span}\{v_1, v_2\}$ as well

EXAMPLE Show $v = (-1, 3, -3)$ is a linear combination of $v_1 = (1, 2, -1)$ and $v_2 = (3, 1, 1)$

$$v = c_1 v_1 + c_2 v_2$$

yields a linear system for $c = (c_1, c_2)$

$$c_1 = 2 \qquad c_2 = -1$$

EXAMPLE $V = P_2$. Express $v(x) = 6x^2 + 12x + 6$ as a linear combination of $v_1(x) = 2x^2 + x + 2$, $v_2(x) = x^2 - x + 1$, <u>if possible</u>

$$v = c_1 v_1 + c_2 v_2 \qquad \text{(equate powers of } x\text{)}$$

Leads to

$$\begin{array}{cc} \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 6 \end{bmatrix} & \begin{matrix} x^2 \\ x^1 \\ x^0 \end{matrix} \end{array}$$

Consistent system: $c_1 = 6$, $c_2 = -6$

EXAMPLE Let $V = \mathbb{R}^3$ and $W \equiv \text{span}\{v_1, v_2, v_3\}$ where

$$v_1 = (1,1,2) \qquad v_2 = (1,0,1) \qquad v_3 = (2,1,3)$$

Does $W = V$?

The question is, can every $b \in V$ be written as a linear combination of v_k :

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = b$$

As a system of equations (in matrix form)

$$(1) \qquad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

matrix A has v_K as columns

The system $Ac = b$ has a solution for all $b \in \mathbb{R}^3$ only if A is invertible. But

$$\det A = 0$$

Since $N(A) \neq \{0\}$, the column space is not $\mathbb{R}^3$
One can show

$$A \sim \begin{array}{c} \begin{array}{ccc} 1 & 2 & \\ \downarrow & \downarrow & \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Columns 1+2 of original matrix for basis

$$\text{col}(A) = \text{span}\{v_1, v_2\}$$

Linear Independence

Definition: $S = \{v_1, v_2 \ldots v_r\} \subset V$ is linearly independent if

$$c_1 v_1 + c_2 v_2 + \cdots c_r v_r = \vec{0}$$

implies $c_k = 0$ for all $k = 1, 2 \ldots r$.

Remark span(S) is always a subspace of V but many vectors in S may be redundant in the sense that if you remove one (or two) the "size" of span(S') is unchanged. Linear indepence is a concept meant to measure this "redundancy". If S is linearly independent then it is in some sense the "smallest" set needed to describe $W = \text{span}(S)$.

EXAMPLE $V = \mathbb{R}^2$ $\quad v_1 = (1, 2)$ $\quad v_2 = (3, 6)$

$S \equiv \{v_1, v_2\}$ dependent

$S' \equiv \{v_1\}$ independent

Here dependence is easily seen since $v_2 = 3v_1$, or

$$c_1 v_1 + c_2 v_2 = \vec{0}$$

if $\quad c_1 = 3 \quad c_2 = -1 \quad$ (non zero!)

Also note span(S) = span(S').

EXAMPLE Show the following set is linearly independent

$$S = \{v_1, v_2, v_3\} \qquad \vec{v}_1 = (1,1,0) \quad \vec{v}_2 = (1,0,-1) \quad \vec{v}_3 = (1,1,1)$$

Recall S is independent if

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0} \qquad \Rightarrow c_k = 0$$

This is equivalent to the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \qquad \vec{c} \quad = \vec{0}$$

$\vec{c} = \vec{0}$ only if A is invertible.

$$\det A = -1$$

hence S is independent.

EXAMPLE $V = \mathbb{R}^n$, $S = \{\vec{v}_1, \dots \vec{v}_r\}$ generalization of previous example

$$c_1 \vec{v}_1 + \cdots c_r \vec{v}_r = \vec{0}$$

$$A\vec{c} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_r \\ | & & | \end{bmatrix} \vec{c} = 0 \qquad A \in \mathbb{R}^{n \times r}$$

Hence $\quad S$ independent $\Leftrightarrow N(A) = \{\vec{0}\}$

EXAMPLE $V = P_2$, $S = \{p_1, p_2, p_3\}$ where

$$p_1 = 2x$$

$$p_2 = x^2 + 1$$

$$p_3 = 2x^2 - 2x + 2$$

Linear independence

(1) $$c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) = 0 \qquad \forall x$$

implies $c_k = 0$. Using p_k definitions (1) can be written.

$$(c_2 + 2c_3)x^2 + (2c_1 - 2c_3)x + (c_2 + 2c_3) = 0$$

Coefficients must vanish

$$A\vec{c} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{0}$$

Since $\det(A) = 0$ (repeated rows), $N(A) \neq \{\vec{0}\}$ so the set is <u>dependent</u>

Can show $A \sim \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$ so that p_3 is a linear combo of p_1, p_2:

$$p_3 = 2p_2 - p_1$$

Thus

$$\text{span}\{p_1, p_2, p_3\} = \text{span}\{p_1, p_2\}$$

Linear Independence of Functions

Theorem Let $V = C^{n-1}(I)$ and $S = \{u_1, \dots u_n\} \subset V$. S is independent on I if the Wronskian

$$W(x) = \begin{vmatrix} u_1 & u_2 & \dots & u_n \\ u_1' & u_2' & \dots & u_n' \\ \vdots & & & \vdots \\ u_1^{(n-1)} & \dots & & u_n^{(n-1)} \end{vmatrix} \neq 0 \quad \forall x \in I$$

Pf: Do in class.

EXAMPLE $u_1 = x^2+x \qquad u_2 = x-1 \qquad u_3 = x^2+3x-2$

$$W(x) = \begin{vmatrix} x^2+x & x-1 & x^2+3x-2 \\ 2x+1 & 1 & 2x+3 \\ 2 & 0 & 2 \end{vmatrix} = 0$$

Conclude $S = \{u_1, u_2, u_3\}$ is a dependent set.

EXAMPLE For what $\lambda \in \mathbb{R}$ are the following independent

$$u_1 = x^2+x+1 \qquad u_2 = x-\lambda \qquad u_3 = 2x^2-\lambda+1$$

After some calculations

$$W(x) = 6\lambda + 2 = 0 \iff \lambda = -\tfrac{1}{3}$$

EXAMPLE $S = \{1, \sin x, \sin 2x\}$

Using addition/mult trig identities

$$W(x) = -3\sin x - \sin 3x$$

Independent on "most" intervals.