Vector Space Bases

Defn Let $S = \{v_1, \dots v_n\} \subset V$ vector space.
The set S is a basis for V if

(i) S is independent

(ii) $V = \text{span}(S)$

Theorem If S is a basis for V then for every $v \in V$ there is a unique set of $c_1, \dots c_n \in \mathbb{R}$ such that

$$v = c_1 v_1 + \cdots c_n v_n$$

Pf/ Suppose $\exists\, c_n, d_n \in \mathbb{R}$ such that

$$v = c_1 v_1 + \cdots c_n v_n$$

$$v = d_1 v_1 + \cdots d_n v_n$$

Subtracting these we find

$$0 = (c_1 - d_1) v_1 + \cdots (c_n - d_n) v_n$$

By independence of v_k the coefficients above must vanish. Equivalently

$$c_k = d_k \qquad \forall k = 1, \dots n \qquad \square$$

Standard Bases

1) For $V = \mathbb{R}^n$ the standard basis E is:

$$E = \{e_1, \dots e_n\}$$

$$e_i = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th} \text{ spot}$$

2) For $V = P_n$ the standard basis E is

$$E = \{p_0, p_1, \dots p_n\}$$

$$E = \{1, x, x^2 \dots x^n\}$$

consequently $p_k(x) \equiv x^k$; $k = 0, 1 \dots n$.

3) For $V = M_{nn}$ the standard basis

$$E = \{e_{ij} : 1 \leq i, j \leq n\}$$

where all elements of the matrix e_{ij} are zero except the (i,j) element which is one.

For example, for M_{22} the basis vectors are

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Coordinates relative to basis S

Defn Let $S = \{v_1, \dots v_n\}$ be a basis for V. Then for every $v \in V$ the is a unique

$$c = (c_1, \dots c_n) \quad \in \mathbb{R}^n$$

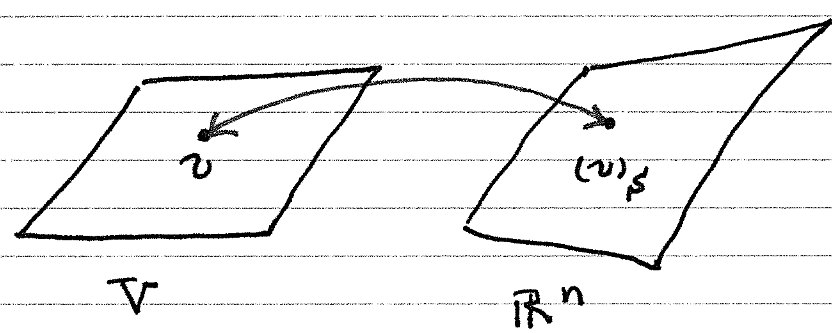
such that

$$v = c_1 v_1 + \cdots \cdot c_n v_n$$

c is the coordinate of v relative to S which we denote

$$c = (v)_S \qquad (1)$$

Remark: Eqn (1) defines a 1-1 correspondence between $v \in V$ and coordinates $(v)_S \in \mathbb{R}^n$



EX $V = \mathbb{R}^3$ standard basis. Let $v \in (a, b, c)$
Then

$$v = ae_1 + be_2 + ce_3 = a\hat{i} + b\hat{j} + c\hat{k}$$

Thus $(v)_E = (a, b, c)$ as well.

EX $V = P_2$ standard basis. Let $v(x) = 7x^2 + 2x$.

$$(v)_E = (0, 2, 7)$$

EX $V = M_{22}$ standard basis. Let $v = \begin{bmatrix} 3 & 2 \\ 0 & 7 \end{bmatrix}$

$$c = (c_1, c_2, c_3, c_4) \in \mathbb{R}^4$$

$$E = S = \{ e_{11}, e_{12}, e_{21}, e_{22} \} = \{ v_1, v_2, v_3, v_4 \}$$

If

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4$$

then

$$(v)_E = (3, 2, 0, 7)$$

is the coordinate of v relative to E

EX Let $V = \mathbb{R}^2$ and $S = \{v_1, v_2\}$ where

$$v_1 = (1, 1) \qquad v_2 = (-1, 1)$$

Find $(v)_S$ where $v = (1, 3)$.

Must find c_1, c_2 such that

$$(1) \qquad c_1 v_1 + c_2 v_2 = v$$

Equivalently this can be written

$$(2) \qquad c_1 v_1 + c_2 v_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

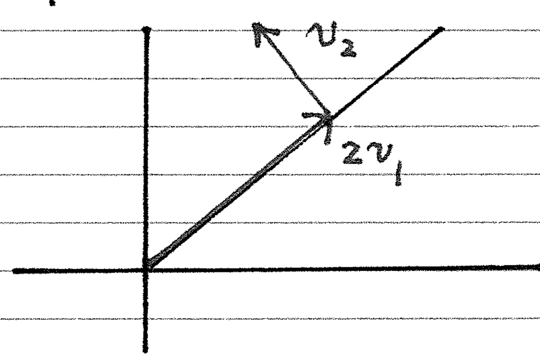
Solving (2) we find $c_1 = 2$, $c_2 = 1$ so that

$$(v)_S = (2, 1) \qquad \text{coordinate.}$$

Recall this means

$$v = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

which in turn has a graphical interpretation



EX Let $V = P_2$ and $S = \{v_1, v_2, v_3\}$ where

$$v_1(x) = x \qquad v_2(x) = x^2 + 3 \qquad v_3(x) = 2x + 1$$

Find $(v)_S$ if $v(x) = 2x^2 + x$.

For all x we must have

$$c_1 x + c_2(x^2+3) + c_3(2x+1) = 2x^2 + x \qquad (1)$$

Written as system (match powers of x)

$$\begin{array}{c} x^2 \\ x \\ 1 \end{array} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Solving this system $c_1 = 13$, $c_2 = 2$, $c_3 = -6$
Hence we conclude

$$(v)_S = (13, 2, -6)$$

EX Let $V = M_{22}$, $S = \{v_1, v_2, v_3, v_4\}$ where

$$v_1 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Given $(v)_S = (1,1,1,1)$, what is v?

$$v = v_1 + v_2 + v_3 + v_4$$

$$v = \begin{bmatrix} 3 & 3 \\ 5 & 4 \end{bmatrix}$$

Dimension

Defn: A nonempty vector space is finite dimensional if it has a basis with a finite number of elements.

Theorem: All bases of finite dimensional vector spaces have the same number of elements.

Defn: Let V be a finite dimensional vector space and S be any basis

$$\dim V \equiv \#\text{vectors in } S$$

Related Theorems

Theorem (Plus Thm) Let $S = \{v_1, \dots v_r\}$

$$v \notin \text{span}(S),\ S \text{ ind} \Rightarrow S^+ = S \cup \{v\} \text{ ind.}$$

Pf/ Let

$$c_1 v_1 + \cdots c_r v_r + c_{r+1} v_{r+1} = 0$$

Must show $c_k = 0$, $\forall k$. Must have $c_{r+1} = 0$ else $v \in \text{span}(S)$ lin. comb. But S ind $\Rightarrow$ $c_1 = \cdots c_r = 0$ hence S^+ independent $\square$

Theorem (Minus)

Let $S \equiv \{v_1, v_2, \ldots v_{r-1}, v\}$. Define $S' = S - \{v\}$ and assume v is a linear combination of $v_1, \ldots v_{r-1}$. Then

$$\text{span}(S) = \text{span}(S')$$

Pf/ Clearly $\text{span}(S') \subset \text{span}(S)$. Must show $\text{span}(S) \subset \text{span}(S')$.

Let $w \in \text{span}(S)$. $\exists k_r \ni$

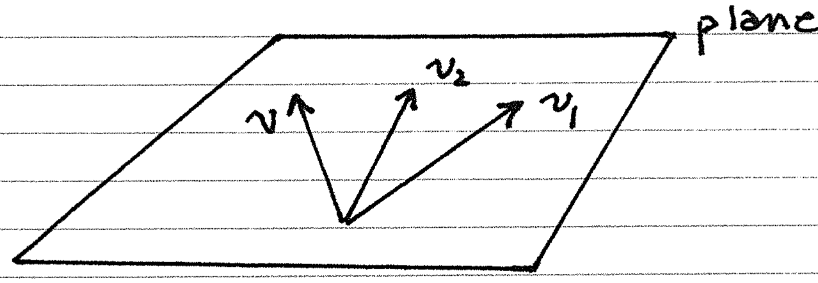
$$(1) \qquad w = k_1 v_1 + k_2 v_2 + \cdots k_{r-1} v_{r-1} + k_r v$$

$$(2) \qquad v = c_1 v_1 + c_2 v_2 + \cdots c_{r-1} v_{r-1} \qquad \text{(hypoth)}$$

Substitute (2) into (1) yields $d_k \ni$

$$w = \sum_{k=1}^{r-1} d_k v_k \in \text{span}(S') \qquad \square$$

A simple picture for this is $V = \mathbb{R}^3$



$$\text{span}(S) = \text{span}(S')$$

Both are the indicated planes.

Miscellaneous Thms (without proof)

Let V be finitedimensional.

Theorem W a subspace of V. Then

(i) $\dim W \leq \dim V$

(ii) $\dim W = \dim V \Rightarrow V = W$

Theorem Let

$$S = \{ v_1, v_2, \ldots v_n \}$$

$$S' = \{ v_1', v_2', \ldots v_m \}$$

be bases for V. Then $m = n$.

EXAMPLE

$$\dim \mathbb{R}^n = n$$

$$\dim P_n = n+1$$

$$\dim M_{nm} = mn$$

EXAMPLE W line thru origin in direction $u=(1,2,3)$

$$W = \text{span}\{v_1\} \qquad v_1 = u$$

$$\dim W = 1$$

EXAMPLE $V = \mathbb{R}^3$ and

$$W \equiv \{x \in V : x_1 + x_2 - x_3 = 0\}$$

$x \in W \Leftrightarrow$

$$x = (x_1, x_2, x_1 + x_2)$$

$$x = x_1 \vec{v}_1 + x_2 \vec{v}_2$$

where $\vec{v}_1 = (1,0,1)$ and $\vec{v}_2 = (0,1,1)$.

Clearly $\vec{v}_K$ are independent and

$$W = \text{span}\{v_1, v_2\} = \text{span}(S)$$

hence S is a basis for W, $\dim W = 2$.

Ex what is the dimension of the solution space

(1) $$3x_1 + x_2 + x_3 + x_4 = 0$$

(2) $$5x_1 - x_2 + x_3 - x_4 = 0$$

This is the same as finding dim N(A) where

$$A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

Since $x_3 = s$, $x_4 = t$ are free; backsolve (1)-(2)

$$x = s \begin{pmatrix} -1/4 \\ -1/4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = s\vec{v}_1 + t\vec{v}_2$$

Hence

$$N(A) = \text{span}\{\vec{v}_1, \vec{v}_2\}$$

$$\dim N(A) = 2$$

Coordinates and Coordinate Transformations

We first introduce some terminology valid for any sets $\mathbb{X}, \mathbb{Y}$.

Definition $T : \mathbb{X} \to \mathbb{Y}$ is injective (1-1) if $T(x_1) = T(x_2) \Rightarrow x_1 = x_2$.

Definition $T : \mathbb{X} \to \mathbb{Y}$ is surjective (onto) if $\forall y \in \mathbb{Y}\ \exists x \in \mathbb{X}$ s.t. $T(x) = y$.

Definition $T : \mathbb{X} \to \mathbb{Y}$ is bijective if it is injective and surjective.

EXAMPLE $\mathbb{X} = [0, \infty)$, $\mathbb{Y} = \mathbb{R}$, $T(x) \equiv x^2$

is not injective (1-1) $\quad T(1) = T(-1) = 1$

is not surjective $\quad T(x) \geq 0$

EXAMPLE $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x) \equiv Ax = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} x$$

Clearly $\det A = 0$ and

$$R(T) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

$R(T) \neq \mathbb{R}^2 \Rightarrow T$ not onto. Also T is not 1-1

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = T\begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$$

Let V be a vector space, $\dim(V) = n$ and

$$S = \{v_1, v_2, \ldots v_n\}$$

be a basis for S.

$$v \longrightarrow (v)_S$$

$$T(v) \equiv (v)_S$$

defines a <u>coordinate map relative</u> to S

$$T : V \to \mathbb{R}^n$$

<u>Theorem</u> The map $T(v) \equiv (v)_S$ satisfies

(i) T is bijective

(ii) $T(cv) = c\,T(v) \qquad \forall c \in \mathbb{R}$

(iii) $T(u+v) = T(u) + T(v)$

(iv) $\{v_1, \ldots v_n\}$ ind. $\Leftrightarrow \{(v_1)_S, \ldots (v_n)_S\}$ ind.

(v) $V = \text{span}(S) \Leftrightarrow \mathbb{R}^n = \text{span}\{(v_1)_S, \ldots (v_n)_S\}$

The linearity properties (ii) – (iii) are easy to show.

Pf (i) First by a previous theorem, coordinates are unique so that $T(v) = (v)_S$ is a well defined function. The map is onto since $\forall c = (v)_S$, $v = c_1 v_1 + \cdots v_n c_n \in V$.

To show 1-1, suppose $T(u) = T(v)$. Then $T(u - v) = 0 \Rightarrow u = v$.

Thus, $T(u) = (u)_S$ is 1-1, onto, well defined.

Pf (iv) First, let S be independent

(a) $\quad k_1 v_1 + \cdots k_n v_n = 0 \quad \in V$

$\quad (k_1 v_1 + \cdots k_n v_n)_S = 0 \quad \in \mathbb{R}^n$

(b) $\quad k_1 (v_1)_S + \cdots k_n (v_n)_S = 0 \quad \in \mathbb{R}^n$

Since (a) $\Rightarrow$ (b), S ind $\Rightarrow k_i = 0$ in (b) hence $S_c = \{ (v_1)_S, \ldots (v_n)_S \}$ indep. in $\mathbb{R}^n$.

We omit the proof that if S_c is independent then S is in V.

EXAMPLE $V = P_2$, $S = \{v_1, v_2, v_3\} = \{x^2+x, x^2-x, x\}$

(i) Does $V = \text{span}(S)$

(ii) Is S independent

The point of this problem is to show we can deteimine if S is a basis for V by seeing if the coordinates (v_k) are a basis for $\mathbb{R}^3$

$E = \{1, x, x^2\}$ standard basis

$(v_1)_E = (0, 1, 1)$

$(v_2)_E = (0, -1, 1)$

$(v_3)_E = (0, 1, 0)$

Want to know if $S_c = \{(v_1)_E, (v_2)_E, (v_3)_E\}$ basis in $\mathbb{R}^3$

Clearly, $\text{span}(S_c) \neq \mathbb{R}^3 \Rightarrow \text{span}(S) \neq V$.
For instance $(1,0,0) \notin \text{span}(S_c)$.

To check independence

$$A \equiv \begin{bmatrix} | & | & | \\ (v_1)_E & (v_2)_E & (v_3)_E \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\det A = 0 \Rightarrow S_c$ dependent : $(v_3)_E = \frac{1}{2}((v_1)_E - (v_2)_E)$.
hence S dependent.

Changing Coordinates

Let S, S' be different bases for V

$$S = \{v_1, v_2, \dots v_n\}$$

$$S' = \{u_1, u_2, \dots u_n\}$$

Then every $v \in V$ has a unique coordinate relative to S and to S'

$$(v)_S = (c_1, c_2 \dots c_n) \qquad \in \mathbb{R}^n$$

$$(v)_{S'} = (d_1, d_2 \dots d_n) \qquad \in \mathbb{R}^n$$

In this setting there is a transition matrix that converts coordinates relative to S to those in S' and vice-versa. For example $\exists A' \in \mathbb{R}^{n \times n}$ s.t.

$$(v)_S = \underbrace{A'}_{\text{transition matrix}} \underbrace{(v)_{S'}}_{\text{in } \mathbb{R}^n}$$

$$(v)_{S'} = A\,(v)_S$$

Textbook notation

$$P_{S' \to S} = A' \qquad \text{convert } S' \text{ to } S$$

$$P_{S \to S'} = A \qquad \text{convert } S \text{ to } S'$$

Transition matrices for $V = \mathbb{R}^n$

Elements of S, S' are in $\mathbb{R}^n$. Let $v \in \mathbb{R}^n$

$$v = c_1 v_1 + \cdots c_n v_n \qquad \text{rel. } S$$

$$v = d_1 u_1 + \cdots d_n u_n \qquad \text{rel. } S'$$

The right sides equal and can be written in matrix form

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

$$B\,(v)_S = B'\,(v)_{S'}$$

The matrices B, B' are both invertible since S, S' are independent.

$$\boxed{(v)_S = B^{-1} B' (v)_{S'}}$$

hence

$$P_{S' \to S} = A' = B^{-1} B'$$

EXAMPLE Find $P_{S\to S'}$ if $V = \mathbb{R}^3$

$$S = \left\{ \begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix} \right\} \qquad S' = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-2\\-1\end{pmatrix} \right\}$$

Then the theory implies

$$B\,(v)_S = B'\,(v)_{S'}$$

$$\begin{bmatrix}1&0&1\\0&1&1\\2&1&2\end{bmatrix}(v)_S = \begin{bmatrix}1&1&1\\1&0&-2\\1&0&-1\end{bmatrix}(v)_{S'}$$

One can calculate B^{-1}

$$B^{-1} = \begin{bmatrix}-1&-1&1\\-2&0&1\\2&1&-1\end{bmatrix}$$

Then

$$P_{S'\to S} = B^{-1}B' = \begin{bmatrix}-1&-1&0\\-1&-2&-3\\2&2&1\end{bmatrix}$$ transition matrix.

Hence $v = (3,-1,0)$ has coordinate $(v)_{S'} = (1,1,1)$.

$$(v)_S = P_{S'\to S}\,(v)_{S'}$$

$$(v)_S = \begin{pmatrix}-2\\-6\\5\end{pmatrix} \Rightarrow \quad v = -2v_1 - 6v_2 + 5v_3$$