Coordinate Transformations when $\dim(V) = 2$

Define two bases for V

$$S = \{v_1, v_2\} \qquad S' = \{u_1, u_2\}$$

Now suppose we know the coordinates of v_k relative to S'

$$(1) \qquad v_1 = a u_1 + b u_2 \qquad (v_1)_{S'} = (a, b)$$

$$(2) \qquad v_2 = c u_1 + d u_2 \qquad (v_2)_{S'} = (a, b)$$

Further suppose we know $(v)_S$:

$$(3) \qquad v = k_1 v_1 + k_2 v_2 \qquad (v)_S = (k_1, k_2)$$

Substitute (1)-(2) into (3)

$$v = k_1(a u_1 + b u_2) + k_2(c u_1 + d u_2)$$

$$v = (k_1 a + k_2 c) u_1 + (k_1 b + k_2 d) u_2$$

give coord. of v rel. to S'

Hence

$$(v)_{S'} = \begin{pmatrix} k_1 a + k_2 c \\ k_1 b + k_2 d \end{pmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

$$\boxed{(v)_{S'} = \underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}}_{P_{S \to S'}} (v)_S}$$

EXAMPLE

$$W \equiv \{ p \in P_2 : p(1) = 0 \} \qquad \dim W = 2$$

Two bases for this subspace of P_2 are

$$S = \{ (x-1), (x-1)^2 \} = \{ v_1, v_2 \}$$

$$S' = \{ x^2 - x, x^2 - 1 \} = \{ u_1, u_2 \}$$

Now find $(v_1)_{S'}$

(1) $$(x-1) = a(x^2 - x) + b(x^2 - 1) \qquad \forall x$$

Since (1) holds $\forall x$, set $x = 0$ to find $b = +1 \Rightarrow a = -1$

$$(v_1)_{S'} = (-1, 1) = (a, b)$$

Next find $(v_2)_{S'}$

(2) $$(x-1)^2 = c(x^2 - x) + d(x^2 - 1)$$

Equating powers of x we find $c = 2, d = -1 \Rightarrow$

$$(v_2)_{S'} = (2, -1)$$

Given the general theory

$$(v)_{S'} = P_{S \to S'} (v)_S$$

wher

$$P_{S \to S'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$