Matrix Fundamental Spaces

$$A = \begin{bmatrix} \text{---} & r_1 & \text{---} \\ & \vdots & \\ \text{---} & r_m & \text{---} \end{bmatrix} = \begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix} \in \mathbb{R}^{m \times n}$$

defines row and column vectors of A.

Definitions

$\text{row}(A)$	$= \text{span}\{r_1, \ldots r_m\}$	row space
$N(A)$	$= \{x : Ax = 0\}$	null space
$\text{col}(A)$	$= \text{span}\{c_1, \ldots c_n\}$	column space
$N(A^T)$	$= \{y : A^T y = 0\}$	null space of A^T

Row Operations

In the following $A \sim B \Rightarrow A$ and B are row equivalent

Theorem Let $A \sim B$. Then

(i) $\text{row}(A) = \text{row}(B)$

(ii) $N(A) = N(B)$

Remark: $A \sim B \not\Rightarrow \text{col}(A) = \text{col}(B)$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad w = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \text{col} A \text{ but } w \notin \text{col} B$$

EXAMPLE Show that

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 2 & 0 & 2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{b}$$

has a solution only if $b_3 = b_1 + b_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 1 & -2 & 1 & b_2 \\ 2 & 0 & 2 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 2 & 0 & 2 & b_1 + b_2 \\ 0 & 0 & 0 & \underbrace{b_3 - b_1 - b_2}_{\substack{\text{must vanish} \\ \text{so } b \in col(A)}} \end{array}\right]$$

$$b_3 = b_1 + b_2$$

So, for instance $\omega = (3, 1, 5)^T \notin col(A)$

Fundamental Spaces (why four)

$row(A)$ *	$row(A^T)$
$col(A)$ *	$col(A^T)$
$N(A)$ *	$N(A^T)$*

But $col(A) = row(A^T)$ and $row(A) = col(A^T)$.
Leaves only four spaces*

Definition

$$rank(A) \equiv \dim row(A)$$

$$nullity(A) \equiv \dim N(A)$$

Theorem Let $A \in \mathbb{R}^{m \times n}$ and $\operatorname{rank}(A) = r \leq m$

(i) $\dim \operatorname{col}(A) = r$

(ii) $\dim N(A) = n - r$

(iii) $\dim N(A^T) = m - r$

Proof (Sketch) Relies on upper echelon form of A

$$A = \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

○ leading one pivots

↓ free variables

$\# \text{ pivots} = r$

$\dim \operatorname{col}(A) = r$

$\# \text{ free variables} = n - r$

$\dim N(A) = n - r$

$\dim N(A^T) = m - r$

<u>Theorem</u> Let

(a) $Ax_0 = b$

(b) $\{v_1, \dots v_k\}$ be a basis for $N(A)$

Then if x solves $Ax = b$

(1) $$x = c_1 v_1 + \cdots c_k v_k + x_0$$

for some $c_i \in \mathbb{R}$. Conversely, if x is given by (1) then it solves $Ax = b$.

<u>Pf $\Rightarrow$</u>

$$Ax_0 = b$$
$$Ax = b$$

$$A(x - x_0) = 0 \quad \Rightarrow \quad x - x_0 \in N(A)$$

<u>Pf $\Leftarrow$</u>

$$\begin{aligned} A(c_1 v_1 + \cdots c_k v_k + x_0) &= c_1 \overset{0}{\cancel{Av_1}} + \cdots c_k \overset{0}{\cancel{Av_k}} + Ax_0 \\ &= Ax_0 \\ &= b \end{aligned}$$

$\square$

<u>Remark</u> (1) defines the general soln of $Ax = b$

$$x = \underbrace{c_1 v_1 + \cdots c_k v_k}_{\text{homogenous soln}} + \underbrace{x_0}_{\text{particular soln}}$$

where $x_0 \in col(A)$ and $x - x_0 \in N(A)$

EXAMPLE   Finding bases of fundamental spaces

$$A = \begin{bmatrix} 4 & 2 & 3 & 1 & 5 \\ 2 & 1 & 1 & 1 & 1 \\ 6 & 3 & 4 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 & 3 & 1 & 5 \\ 0 & 0 & -1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1) Here, rank $A = 2$ and

$$\text{row}(A) = \text{span}\{(4,2,3,1,5), (0,0,-1,1,-3)\} = \text{span}\{r_1, r_2\}$$

2) For $N(A)$ there are three free variables x_2, x_4, x_5. Set these to 0 and 1 and backsolve the upper echelon

$$N(A) = \text{span}\{v_1, v_2, v_3\} \qquad \dim N(A) = 3$$

$$v_1 = (1, 0, -3, 0, 1)$$

$$v_2 = (-1, 0, 1, 1, 0)$$

$$v_3 = (-\tfrac{1}{2}, 1, 0, 0, 0)$$

Note $r_k \perp v_k$, i.e. $v_1 \cdot r_1 = 4 - 9 + 5 = 0$.

3) Pivots are in columns 1, 3 hence columns 1, 3 of A form a basis for col(A)

$$\text{col}(A) = \text{span}\left\{ \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$$