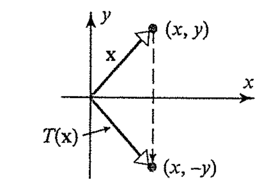
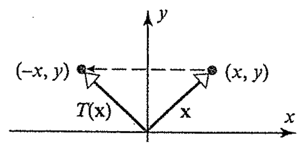
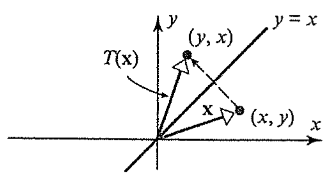
Geometry of Matrix Transformations on $\mathbb{R}^2$

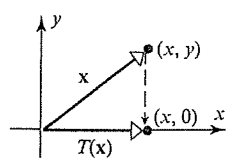
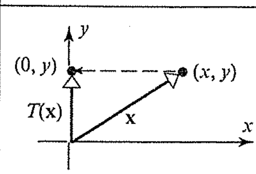
Reflexions

Table 1

Operator	Illustration	Images of e_1 and e_2	Standard Matrix
Reflection about the x-axis $T(x, y) = (x, -y)$		$T(e_1) = T(1, 0) = (1, 0)$ $T(e_2) = T(0, 1) = (0, -1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the y-axis $T(x, y) = (-x, y)$		$T(e_1) = T(1, 0) = (-1, 0)$ $T(e_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the line $y = x$ $T(x, y) = (y, x)$		$T(e_1) = T(1, 0) = (0, 1)$ $T(e_2) = T(0, 1) = (1, 0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

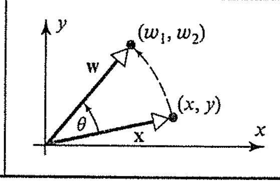
Projections

Table 3

Operator	Illustration	Images of e_1 and e_2	Standard Matrix
Orthogonal projection onto the x-axis $T(x, y) = (x, 0)$		$T(e_1) = T(1, 0) = (1, 0)$ $T(e_2) = T(0, 1) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection onto the y-axis $T(x, y) = (0, y)$		$T(e_1) = T(1, 0) = (0, 0)$ $T(e_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

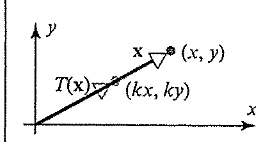
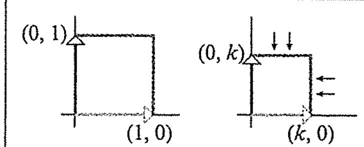
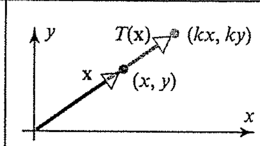
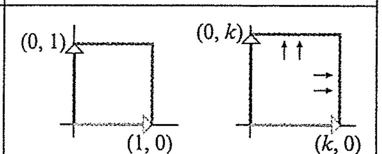
Rotation

Table 5

Operator	Illustration	Rotation Equations	Standard Matrix
Counterclockwise rotation about the origin through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

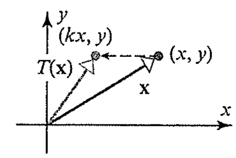
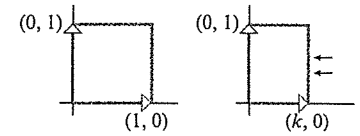
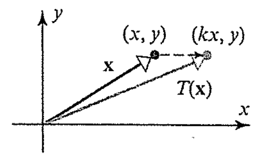
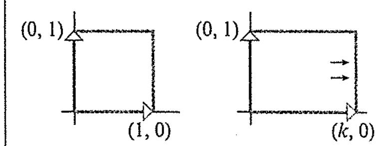
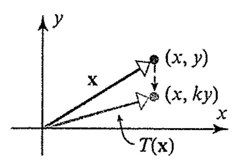
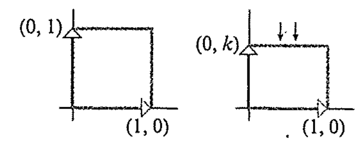
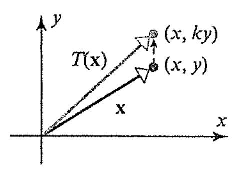
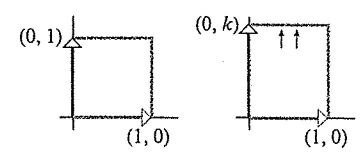
Contraction and Dilation

Table 7

Operator	Illustration $T(x, y) = (kx, ky)$	Effect on the Unit Square	Standard Matrix
Contraction with factor k in R^2 ($0 \le k < 1$)			$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Dilation with factor k in R^2 ($k > 1$)			

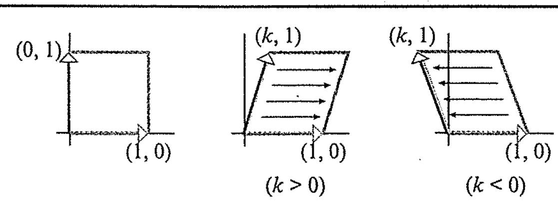
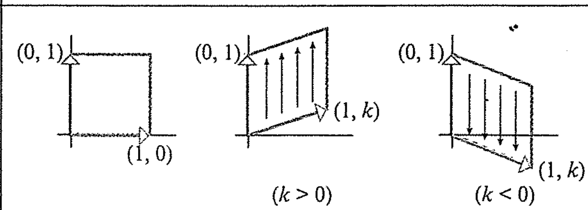
Compression and Expansions

Table 9

Operator	Illustration $T(x, y) = (kx, y)$	Effect on the Unit Square	Standard Matrix
Compression in the x-direction with factor k in R^2 $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Expansion in the x-direction with factor k in R^2 $(k > 1)$			
Operator	**Illustration $T(x, y) = (x, ky)$**	**Effect on the Unit Square**	**Standard Matrix**
Compression in the y-direction with factor k in R^2 $(0 \le k < 1)$			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Expansion in the y-direction with factor k in R^2 $(k > 1)$			

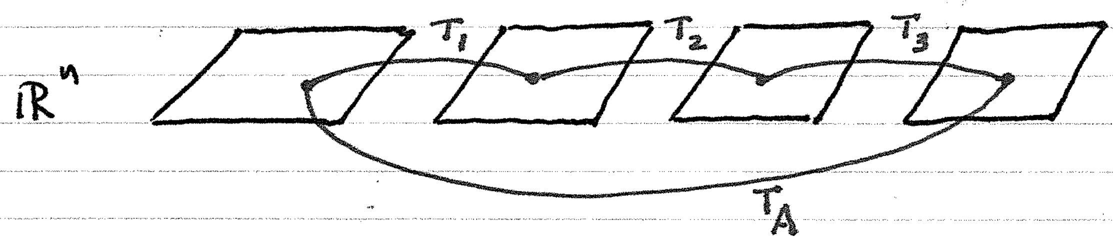
Shear

Table 10

Operator	Effect on the Unit Square	Standard Matrix
Shear in the x-direction by a factor k in R^2 $T(x, y) = (x + ky, y)$		$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear in the y-direction by a factor k in R^2 $T(x, y) = (x, y + kx)$		$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Compositions of Transformations

$$T_A(x) = (T_3 \circ T_2 \circ T_1)(x) = T_3(T_2(T_1(x)))$$



Since matrix transformations are defined by their matricies, the matrix for T is easily found

$$T_1(x) = A_1 x$$

$$T_2(x) = A_2 x$$

$$T_3(x) = A_3 x$$

$$T(x) = (A_3 A_2 A_1)\,x \equiv Ax$$

For two simple examples we use

$T_1(x) = A_1 x \qquad A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ reflection about $y = x$

$T_2(x) = A_2 x \qquad A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Counterclockwise rotation by $\pi/2$

$T_3(x) = A_3 x \qquad A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ Dilation by factor 2

EXAMPLE $T_A(x) \equiv (T_3 \circ T_1 \circ T_2)(x)$

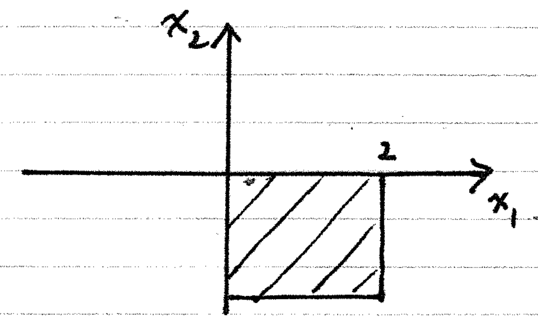


Image of unit square.

$\ast \quad A = A_3 A_1 A_2 = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

EXAMPLE $T_B(x) \equiv (T_3 \circ T_2 \circ T_1)(x)$

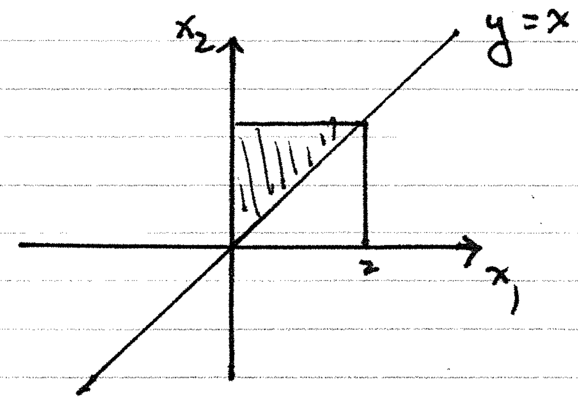


Image of $[0,1]^2$

$\ast \quad B = B_3 B_2 B_1 = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

Note the sole difference in these two examples is the order of the operations.

Not surprisingly, since matrix multiplication doesn't always commute, $A \neq B$.