

Vector Space Definition Let V be a nonempty set on which the operations of addition $+$ and scalar multiplication have been defined:

- (i) $\mathbf{u} + \mathbf{v}$ is defined $\forall \mathbf{u}, \mathbf{v} \in V$
- (ii) $c\mathbf{u}$ is defined $\forall \mathbf{u} \in V, \forall c \in \mathbb{R}$.

The set V is called a vector space if additionally, $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\forall b, c \in \mathbb{R}$ the following axioms hold:

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| (A1) | $\mathbf{u} + \mathbf{v} \in V$ | V closed under addition |
| (A2) | $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | addition is commutative |
| (A3) | $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ | addition is associative |
| (A4) | $\exists \mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ | existence of a zero vector |
| (A5) | $\exists -\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ | existence of a negative element |
| (A6) | $c\mathbf{u} \in V$ | closed under scalar multiplication |
| (A7) | $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ | distributive property I |
| (A8) | $(b + c)\mathbf{u} = b\mathbf{u} + c\mathbf{u}$ | distributive property II |
| (A9) | $c(\beta\mathbf{u}) = (c\beta)\mathbf{u}$ | commutativity of scalar multiplication |
| (A10) | $1\mathbf{u} = \mathbf{u}$ | scalar multiplication identity element |

Sometimes the symbols \oplus and \odot will be used to denote vector addition and scalar multiplication, respectively. This will only be done when vector addition and scalar multiplication is defined in some (confusing) nontraditional way. Then $\mathbf{u} + \mathbf{v}$ will be written $\mathbf{u} \oplus \mathbf{v}$ and $c\mathbf{u}$ will be written $c \odot \mathbf{u}$. As one example

$$3(\mathbf{u} + \mathbf{v}) = 3 \odot (\mathbf{u} \oplus \mathbf{v})$$