Math 430 Mathematical Biology – Homework 4

Due: Tuesday, April 11, 2023. NAME: ____________________

1) [10] Capasso and Serio (1978) considered the following SIR epidemic model with emigration of the susceptible S:

$$\frac{dS}{dt} = -g(I)S - \lambda S \tag{1}$$
$$\frac{dI}{dt} = g(I)S - \beta I \tag{2}$$
$$\frac{dR}{dt} = \lambda S + \beta I \tag{3}$$

where $g(I) = \alpha I e^{-I}$. The parameters α, β, λ are all **positive**. The function $g(I)$ is meant to take into account "psychological" effects. In particular, when the number of infectives I is large the number of interactions $g(I)S$ is smaller since the susceptibles S notice the infected people and actively try to stay away from them.

a) Is the total population $N = S + I + R$ conserved?

b) Show the (S, I) system has no positive (physical $S > 0, I > 0$) equilibria.

c) Show the $(S, I) = (0, 0)$ extinction state is stable.

d) Create a pplane9 (phase plane) diagram showing that S, I both die out and hence all people recover. Include the nullclines and at least one trajectory illustrating the aforementioned dynamic. You may use $\alpha = \beta = \lambda = 1$ as parameter values.

2) [5] Nondimensionalize the SIR model

$$\frac{dS}{dt} = -\alpha SI \tag{4}$$
$$\frac{dI}{dt} = \alpha SI - \beta I \tag{5}$$
$$\frac{dR}{dt} = \beta I \tag{6}$$

In particular, scale the dependent and independent variables :

$$S = sS^* \quad , \quad I = iI^* \quad , \quad R = rR^* \quad , \quad t = \tau t^* \quad ,$$

and show that for a certain choice of the constants S^*, I^*, R^* and t^* the resulting system for lower case s, i, r, τ contains <u>no parameters</u>.

The new dimensionless system should look like:

$$\frac{ds}{d\tau} = -si \tag{7}$$

$$\frac{di}{d\tau} = si - i \tag{8}$$

$$\frac{dr}{d\tau} = i \tag{9}$$

State the constants S^*, I^*, R^* and t^* formulae in terms of α and β.

3) [5] Use Law of Mass Action to write out the (molar) concentration differential equations for the following reactions.

$$X + 2Y \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} Y + Z$$