

Math 430 Mathematical Biology

Name: _____

Final - Takehome (max=50)

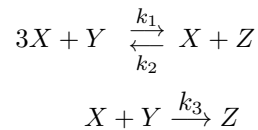
Due Thursday May 11, 2023.

12:00-1:50pm in class - Wil 1-130

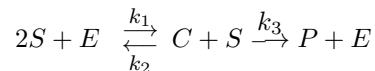
Instructions : The following guidelines must be observed:

- a) You may use the textbook, notes on the course website, or your own classroom notes
- b) You must work alone and may not talk to fellow classmates. You may ask me clarifying questions.
- c) Merely stating an answer is insufficient. You must show your work.
- d) This test (with your name on it) must be stapled to your work.
- e) You may use matlab and/or a calculator.

1. [10] Use Law of Mass Action to write out the (molar) concentration differential equations for the following reactions.



2) [10] Another variation of Michaelis-Menten kinetics has the reaction equations:



Conservation of receptors imply

$$E = N - C$$

where N is the total number of receptors. The quasi-steady state assumption implies

$$\frac{dC}{dt} = k_1 S^2 E - (k_2 + k_3)CS = 0$$

Use these two equations to derive the production rate $\frac{dP}{dt}$ in terms of the substrate concentration S only. Specifically, find a function $V(S)$ such that

$$\frac{dP}{dt} = k_3 CS = V(S)$$

3. [10] A nutrient $c(x, t)$ diffuses in its domain $x \in [1, 2]$. Consequently, $c(x, t)$ satisfies the partial differential equation:

$$c_t = Dc_{xx}$$

The concentration has the following boundary conditions:

$$\begin{aligned} c(1, t) &= 4 \\ c(2, t) + c_x(2, t) &= 6 \end{aligned}$$

Find a formula for the steady state $\bar{c}(x)$ satisfying the same boundary conditions (at $x = 1$ and $x = 2$) and

$$0 = D\bar{c}_{xx}$$

4. [20] Baleen whales (population Y) eat krill (population X). Both krill and whales have density dependent growth rates. And, their respective carrying capacities are K and $K_w = dX$. The resulting dimensional model equations are:

$$\frac{dX}{dt} = aX \left(1 - \frac{X}{K}\right) - bXY \quad (1)$$

$$\frac{dY}{dt} = cY \left(1 - \frac{Y}{dX}\right) \quad (2)$$

where a, b, c, d, K are all constant.

a) Nondimensionalize the model equations using the definitions

$$x = \frac{X}{X^*} \quad y = \frac{Y}{Y^*} \quad \tau = \frac{t}{t^*}$$

so the dimensionless system is:

$$\frac{dx}{d\tau} = \alpha x(1-x) - xy \quad (3)$$

$$\frac{dy}{d\tau} = \beta y \left(1 - \frac{y}{x}\right) \quad , \quad x \neq 0 \quad (4)$$

Summarize what each of X^*, Y^*, t^*, α and β equal in terms of the original parameters (a, b, c, d, K) .

- b) Show that if $\alpha > 0$, the model has a (physical strictly positive) coexistence equilibria $P_1(\bar{x}, \bar{y})$. Find the \bar{x} and \bar{y} components of the equilibria.
- c) Show the coexistence equilibria $P_1(\bar{x}, \bar{y})$ is stable. There is a lot of algebra in this one but the determinant of the Jacobian greatly simplifies!
- d) What is \bar{x} in the whale extinction state $P_0(\bar{x}, 0)$? Is the extinction state stable?