Math 430 Mathematical Biology Name: ______________________
Final - Takehome (max=50)
Due Thursday May 11, 2023.
12:00-1:50pm in class - Wil 1-130

Instructions : The following guidelines must be observed:

a) You may use the textbook, notes on the course website, or your own classroom notes

b) You must work alone and may not talk to fellow classmates. You may ask me clarifying questions.

c) Merely stating an answer is insufficient. You must show your work.

d) This test (with your name on it) must be stapled to your work.

e) You may use matlab and/or a calculator.

---

1. [10] Use Law of Mass Action to write out the (molar) concentration differential equations for the following reactions.

$$3X + Y \underset{k_2}{\overset{k_1}{\rightleftarrows}} X + Z$$

$$X + Y \xrightarrow{k_3} Z$$

2) [10] Another variation of Michaelis-Menten kinetics has the reaction equations:

$$2S + E \underset{k_2}{\overset{k_1}{\rightleftarrows}} C + S \xrightarrow{k_3} P + E$$

Conservation of receptors imply

$$E = N - C$$

where N is the total number of receptors. The quasi-steady state assumption implies

$$\frac{dC}{dt} = k_1 S^2 E - (k_2 + k_3)CS = 0$$

Use these two equations to derive the production rate $\frac{dP}{dt}$ in terms of the substrate concentration S **only**. Specifically, find a function $V(S)$ such that

$$\frac{dP}{dt} = k_3 CS = V(S)$$

3. [10] A nutrient $c(x,t)$ diffuses in its domain $x \in [1,2]$. Consequently, $c(x,t)$ satisfies the partial differential equation:

$$c_t = Dc_{xx}$$

The concentration has the following boundary conditions:

$$\begin{aligned} c(1,t) &= 4 \\ c(2,t) + c_x(2,t) &= 6 \end{aligned}$$

Find a formula for the steady state $\bar{c}(x)$ satisfying the same boundary conditions (at $x = 1$ and $x = 2$) and

$$0 = D\bar{c}_{xx}$$

4. [20] Baleen whales (population Y) eat krill (population X). Both krill and whales have density dependent growth rates. And, their respective carrying capacities are K and $K_w = dX$. The resulting dimensional model equations are:

$$\frac{dX}{dt} = aX\left(1 - \frac{X}{K}\right) - bXY \tag{1}$$

$$\frac{dY}{dt} = cY\left(1 - \frac{Y}{dX}\right) \tag{2}$$

where a, b, c, d, K are all constant.

a) Nondimensionalize the model equations using the definitions

$$x = \frac{X}{X^*} \qquad y = \frac{Y}{Y^*} \qquad \tau = \frac{t}{t^*}$$

so the dimensionless system is:

$$\frac{dx}{d\tau} = \alpha x\,(1 - x) - xy \tag{3}$$

$$\frac{dy}{d\tau} = \beta y\left(1 - \frac{y}{x}\right) \quad , \quad x \neq 0 \tag{4}$$

Summarize what each of X^*, Y^*, t^*, α and β equal in terms of the original parameters (a, b, c, d, K).

b) Show that if $\alpha > 0$, the model has a (physical strictly positive) coexistence equilibria $P_1(\bar{x}, \bar{y})$. Find the $\bar{x}$ and $\bar{y}$ components of the equilibria.

c) Show the coexistence equilibria $P_1(\bar{x}, \bar{y})$ is stable. There is a lot of algebra in this one but the determinant of the Jacobian greatly simplifies!

d) What is $\bar{x}$ in the whale extinction state $P_0(\bar{x}, 0)$? Is the extinction state stable?