Math 430 Mathematical Biology Name: ______________________

Midterm - Takehome (max=50)

Due Tuesday March 28, 2023.

Instructions : The following guidelines must be observed:

a) You may use the textbook, notes on the course website, or your own classroom notes

b) You must work alone and may not talk to fellow classmates but you may ask me clarifying questions.

c) Merely stating an answer is insufficient. You must show your work.

d) This test (with your name on it) must be stapled to your work.

e) You may use matlab and/or a calculator.

f) Make sure your work is legible and in the proper order.

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1. [10pts] Recall the simple *growth with harvesting model*

$$x_{n+1} = \lambda x_n - h$$

where x_n is the chicken population after n years, x_0 is the initial population, λ is the growth rate and h is the harvesting rate (chickens per year). The units are in hundreds of chickens so that x_n=5 means 500 chickens. The solution derived in class was:

$$x_n = \lambda^n x_0 - \left(\frac{\lambda^n - 1}{\lambda - 1}\right) h$$

Suppose $x_0 = 10$, $\lambda = 5$ and the population becomes extinct after $n = 10$ years. At what rate h were the chickens harvested?

2. [10pts] Find the general solution of the following difference equation:

$$2x_{n+2} - 3x_{n+1} + x_n = 0$$

What does x_n approach as $n \to \infty$

3. [10pts] A discrete time population of an annual plant has population x_n modelled by:

$$x_{n+1} = f(x_n) = -\alpha x_n \ln x_n \tag{1}$$

where $\alpha > 0$, $f(x) > 0$ on $x \in (0, 1)$.

a) Draw an accurate sketch of $y = x$ and $y = f(x)$ for $0 < x \leq 1$ labelling the positive fixed point $\bar{x}$. You may use the fact that $\lim_{x\to 0+} f(x) = 0$ and choose $\alpha = 1$.

b) Find a formula for the positive fixed point $\bar{x}$ of (1) in terms of α.

c) Compute and then <u>simplify</u> $f'(\bar{x})$ (it really simplifies!).

d) Given your result in c), for what α is the fixed point stable.

4. [10pts] Consider the following nonlinear system:

$$\frac{dx}{dt} = x^3 - y \tag{2}$$
$$\frac{dy}{dt} = y - x \tag{3}$$

a) Draw the x and y nullclines and then label the equilibria (three of them)

b) Compute the Jacobian for the system

c) Compute the Jacobian $A = DF(P_3)$ at the equilibria $P_3 = (1, 1)$

d) Use $Tr(A)$ and $det(A)$ to classify the equilibria P_3, i.e. saddle, stable node, unstable node, stable spiral, unstable spiral, center etc.

5. [10pts] We seek to model a predator-prey system where the prey growth rate is density dependent and the prey is also harvested at a rate proportional to its population (with harvesting parameter h). After nondimensionalizing such a model, we obtain the model equations:

$$\frac{dx}{dt} = f(x, y) = ax(1 - x) - axy - hx$$
$$\frac{dy}{dt} = g(x, y) = -cy + xy$$

Here, all parameters are positive and $a > h$.

a) Find all equilibria

$$\begin{aligned} P_0 &= \textit{both predator and prey extinct} \\ P_1 &= \textit{only predator extinct} \\ P_2 &= \textit{coexistence state} \end{aligned}$$

b) Find an inequality that the harvesting rate h must satisfy for the coexistence state to be stable. You'll need to compute the trace and determinant of $F = (f, g)$ at P_2.

The purpose of b) is to find constraints on the harvesting rate that will make sure we don't over harvest and kill out the ecosystem.