First order linear difference eqn population model

a_n = # females in n^{th} generation
p_n = # progeny in n^{th} generation

f = # progeny per female
m = fractional mortality of progeny
r = fraction of females in total population

Given there definitions

(1) $$p_{n+1} = f\, a_n$$

and

(2) $$a_{n+1} = r \underbrace{(1-m)\, p_{n+1}}_{\text{progeny that survived generation.}}$$

Combining (1)-(2)

$$\boxed{\begin{gathered} a_{n+1} = \lambda\, a_n \\ \lambda \equiv f r (1-m) \end{gathered}}$$

Linear first order solution

$$x_{n+1} = \lambda x_n \qquad n = 0, 1, 2, \ldots$$

Here x_0 is the initial condition

$$x_1 = \lambda x_0$$

$$x_2 = \lambda x_1 = \lambda^2 x_0$$

$$x_3 = \lambda x_2 = \lambda^3 x_0$$

By induction

$$\boxed{x_n = \lambda^n x_0}$$

Solution also works if $\lambda \in \mathbb{C}$ is complex.

$$|\lambda| < 1 \Rightarrow x_n \to 0$$

$$|\lambda| > 1 \Rightarrow x_n \to \infty$$

when $|\lambda| = 1$, x_n is constant or period 2.

Ex Let x_n = number of bacteria after n days

$$x_{n+1} = \lambda x_n$$

what is the doubling time?

$$x_n = \lambda^n x_0$$

In how many days n does $x_n = 2x_0$

$$2 = \lambda^n$$

$$\boxed{n = \frac{\ln 2}{\ln \lambda}}$$

Ex Conversion of time units

x_n = # bact after n days

X_N = # bact after N weeks

If we model x_n as before $x_{n+1} = \lambda x_n$

$$X_1 = x_7$$

$$X_2 = x_{14}$$

$$X_3 = x_{21}$$

Clearly

$$X_{N+1} = \lambda^7 X_N$$

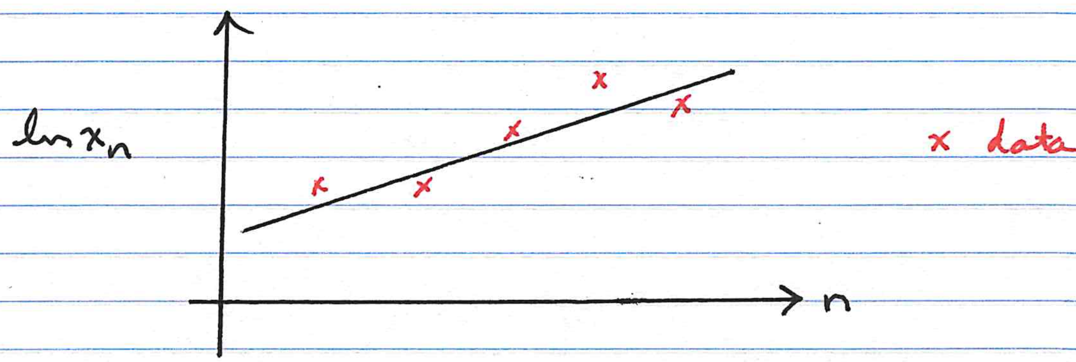
EX Experimental Data

$$x_{n+1} = \lambda x_n$$

$$x_n = \lambda^n x_0$$

From which we derive

$$(1) \qquad \underset{\text{plot}}{\ln x_n} = n \underset{\text{don't know}}{\ln \lambda} + \underset{\text{know}}{\ln x_0}$$



Choose slope $\ln \lambda$ to best match data.

Generally done with <u>linear regression</u>.

Harvesting Model

(1) $$\boxed{x_{n+1} = \lambda x_n - h} \qquad \lambda > 1, \; h \geq 0$$

where

x_n = number of chicken at year n

h = number of chickens harvested per year

To solve (1) we translate x_n into a form we know how to solve

(2) $$x_n = y_n + \bar{x}$$

Substitute (2) into (1)

$$y_{n+1} + \bar{x} = \lambda(y_n + \bar{x}) - h$$

Rearrange terms

$$y_{n+1} = \lambda y_n + \underbrace{(\lambda - 1)\bar{x} - h}_{\text{choose } \bar{x} \text{ so this vanishes}}$$

Thus, for

(3) $$\bar{x} = \frac{h}{(\lambda - 1)}$$

we have

(4) $$y_{n+1} = \lambda y_n$$

This we know how to solve: $y_n = \lambda^n y_0$.

Hence

$$x_n = \lambda^n y_0 + \bar{x}$$

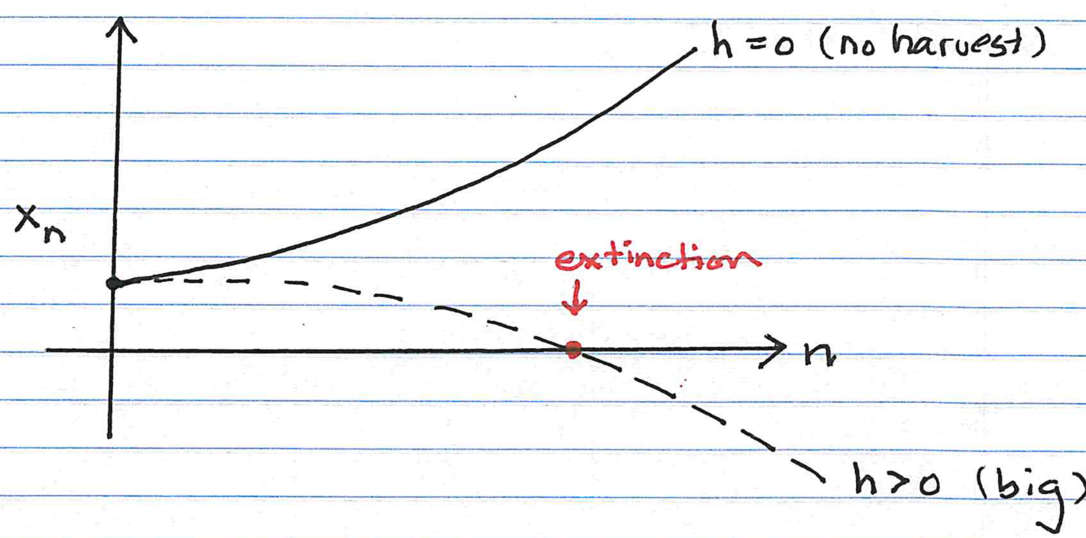
$$x_n = \lambda^n (x_0 - \bar{x}) + \bar{x}$$

Collecting terms and using (3)

$$\boxed{x_n = \lambda^n x_0 - \left(\frac{\lambda^n - 1}{\lambda - 1}\right) h}$$

regular growth

loss to harvesting



QUESTION ONE : For what h does population remain constant?

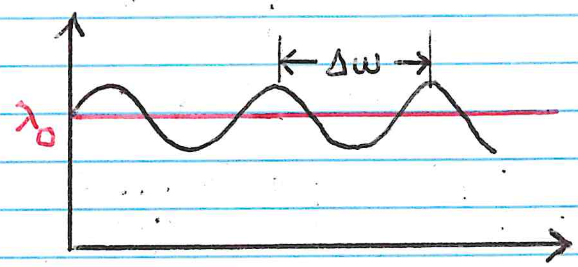
QUESTION TWO : At what year does population become extinct?

Seasonal Growth Rates and Harvesting

$$x_{n+1} = \lambda_n x_n - h$$

growth rate varies with time n.
As an example

$$\lambda_n = \lambda_0 + \alpha \sin(\omega n)$$



Seasonal growth rate

n (months)

Here

$$\omega = \frac{2\pi}{12}$$

gives an annual oscillation frequency $\Delta\omega = 12$

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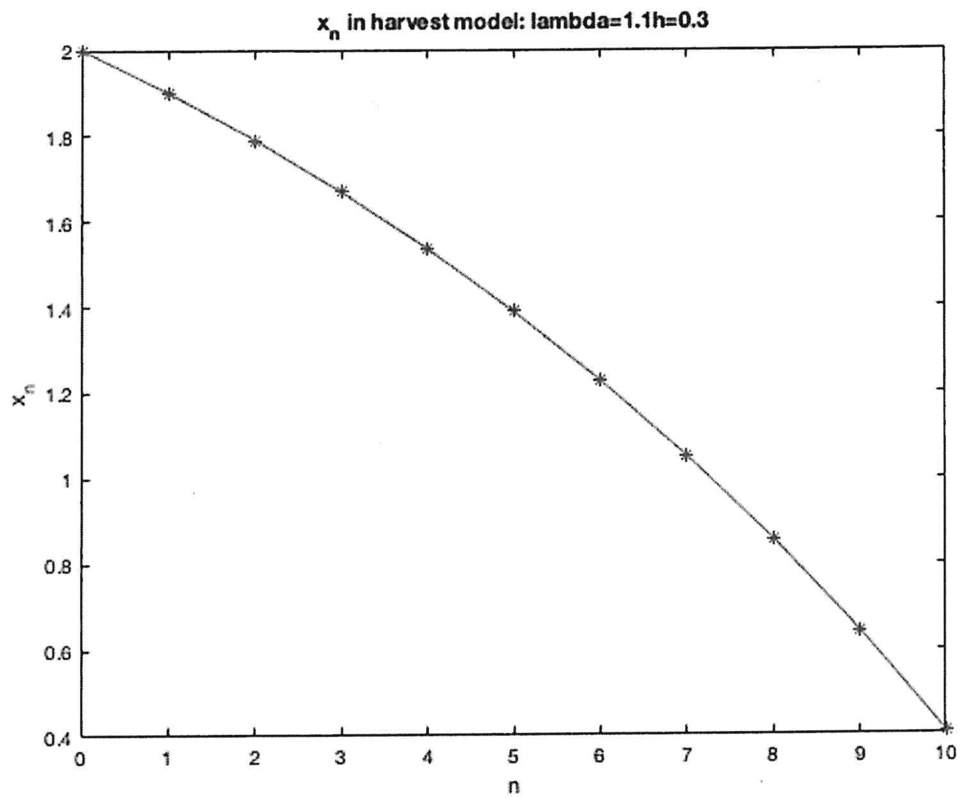
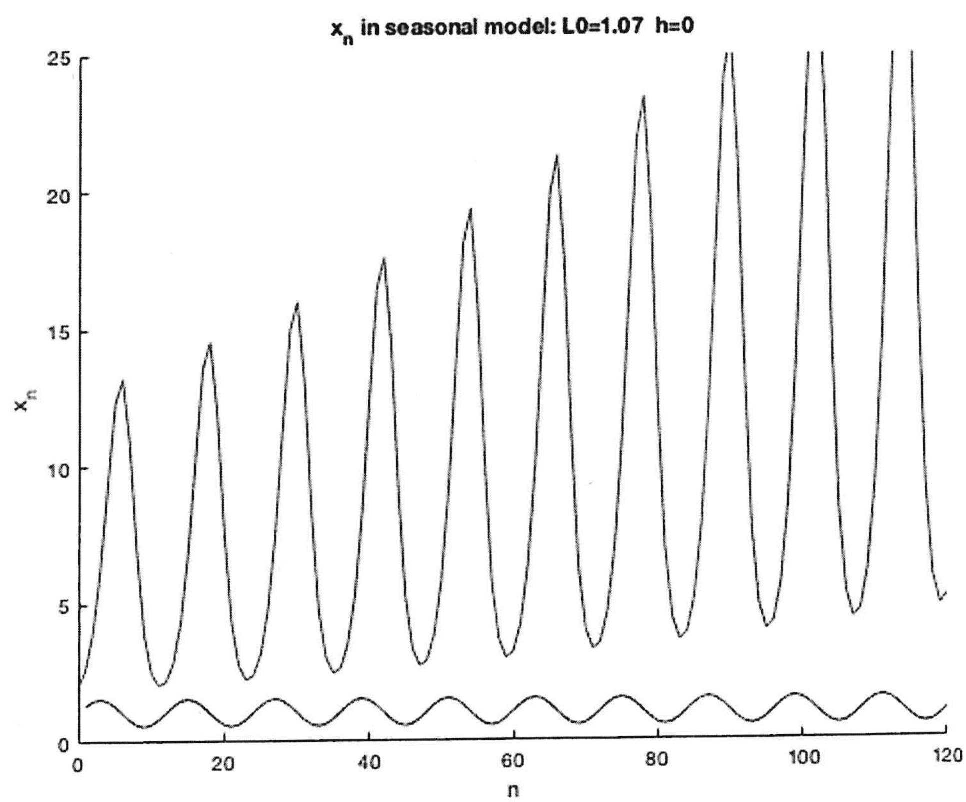
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%      Harvesting model
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%     X(n+1)= lambda X(n)  - h
%
% with initial condition X(0)=X0
% has exact solution:
%
%   X(n)=lambda^n X0 -(lambda^n-1)/(lambda-1) h
%
% First calculate X(n) without the formula above.
%
clear X;
lambda=1.1;
X(1)=2;
%h=0.20;
h=0.30;
N=10;
for n=1:N
    X(n+1)=lambda*X(n)-h;
end;
figure(4)
plot(0:N,X)
%
% Now compute Y=X using formula above
%
m=0:N;
Y=lambda.^m*X(1) -(lambda.^m-1)./(lambda-1)*h;
hold on
plot(m,Y,'r*')
title(['x_n in harvest model:
 lambda=',num2str(lambda),'h=',num2str(h)])
xlabel('n')
ylabel('x_n')
hold off

%
%
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%      Harvesting model with Seasonal Variation
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%     Seasonal variation of growth rate. Suppose
%     n=time (in months) since start growth rate at
%     time n is L(n):
%
%     L = L0+alpha*sin(omega*n)
%
%     X(n+1)= L(n) X(n)  - h
%
```

```
%     with initial condition X(1)=X0.
%
%
clear X;
N=120;
L0=1.07;
alpha=0.5;
omega=2*pi./12;
L=L0+alpha*sin(omega*(1:N));
%
%
X(1)=2;
h=0.000;
%h=0.035;
for n=1:N
    X(n+1)=L(n)*X(n)-h;
end;
%
figure(5)
clf
hold on
plot(0:N,X,'r-')
plot(1:N,L,'b-')
title(['x_n in seasonal model: L0=',num2str(L0),'  h=',num2str(h)])
xlabel('n')
ylabel('x_n')
axis([0,N,0,25])
hold off
```


x_n in harvest model: lambda=1.1h=0.3
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
x_n
0 1 2 3 4 5 6 7 8 9 10
n


x_n in seasonal model: L0=1.07 h=0
25
20
15
10
5
0
x_n
0 20 40 60 80 100 120
n