Nonlinear Density Dependent Models

(1) $$x_{n+1} = x_n \, g(x_n) = f(x_n)$$

where here

$$g(x) = \text{growth rate at density } x$$

Remarks

a) If $g(x) = r$, $r > 0$ constant then the model in (1) is the linear model

$$x_{n+1} = r x_n$$

b) If at some time n we have $g(x_n) = 1$ then

$$x_{n+1} = x_n$$

$$x_{n+2} = x_{n+1} \, g(x_{n+1}) = x_n$$

$$x_{n+3} = x_{n+2} \, g(x_{n+2}) = x_n$$

In other words, if $g(x_n) = 1$ then

$$x_N = x_n \qquad \forall N \geq n$$

Note "$\forall$" means for all.

An orbit of $x_{n+1} = f(x_n)$

$$\gamma(x_0) = \{x_0, x_1, x_2, x_3, \ldots\}$$

$$\gamma(x_0) = \{x_0, f(x_0), f^2(x_0), f^3(x_0), \ldots\}$$

gives discrete time series for population given an initial density x_0.

EXAMPLE $f(x) = x(1-x)$ $x_0 = \frac{1}{2}$

$$x_1 = f(x_0) = \frac{1}{2}\left(1-\frac{1}{2}\right) = \frac{1}{4}$$

$$x_2 = f(x_1) = \frac{1}{4}\left(1-\frac{1}{4}\right) = \frac{3}{16}$$

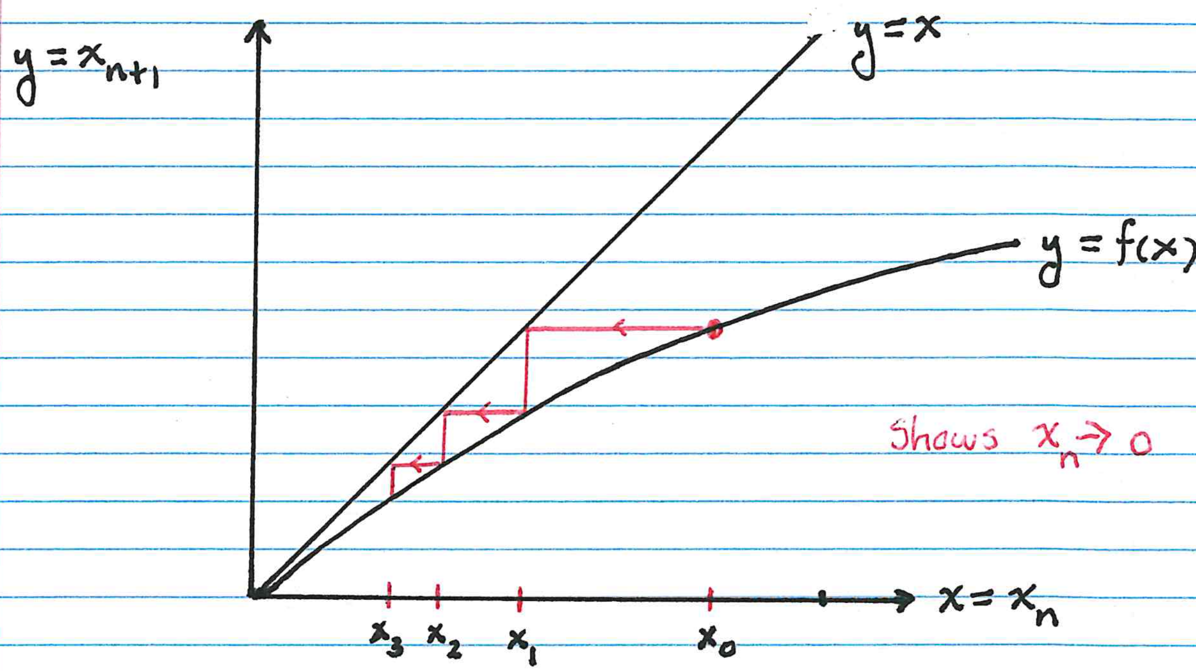
$$x_3 = f(x_2) = \frac{3}{16}\left(1-\frac{3}{16}\right) = \frac{39}{256}$$

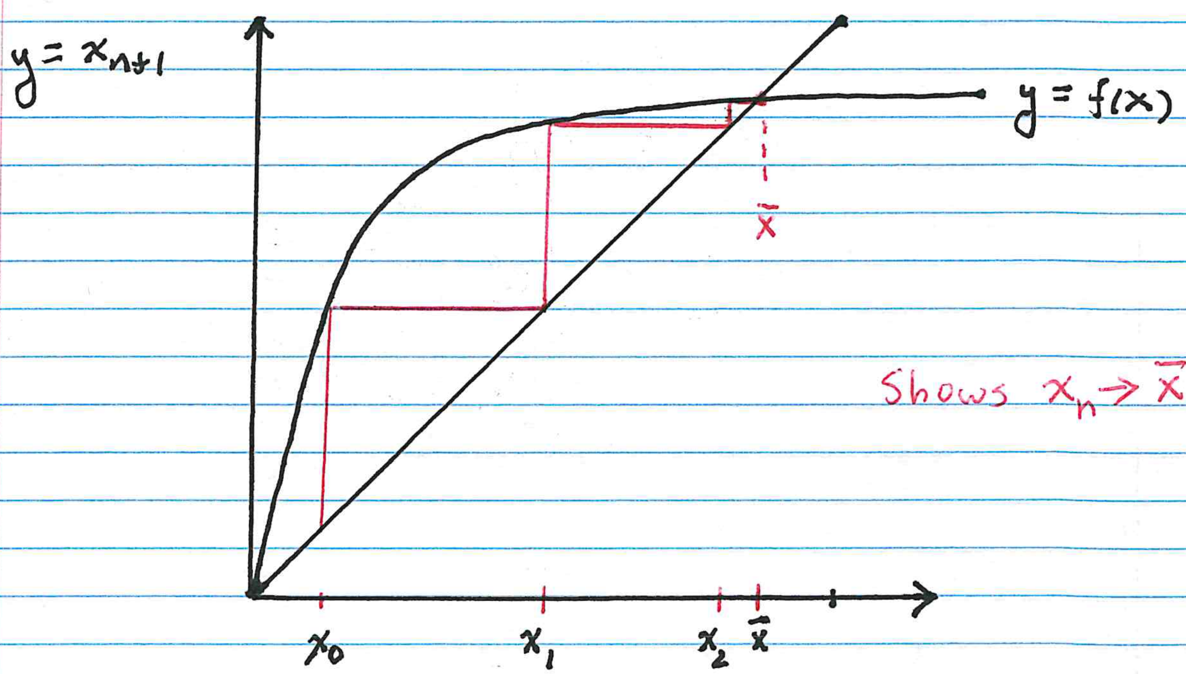
Hence

$$\gamma(x_0) = \left\{\frac{1}{2}, \frac{1}{4}, \frac{3}{16}, \frac{39}{256}, \ldots\right\}$$

Cobwebs $\quad x_{n+1} = f(x_n)$

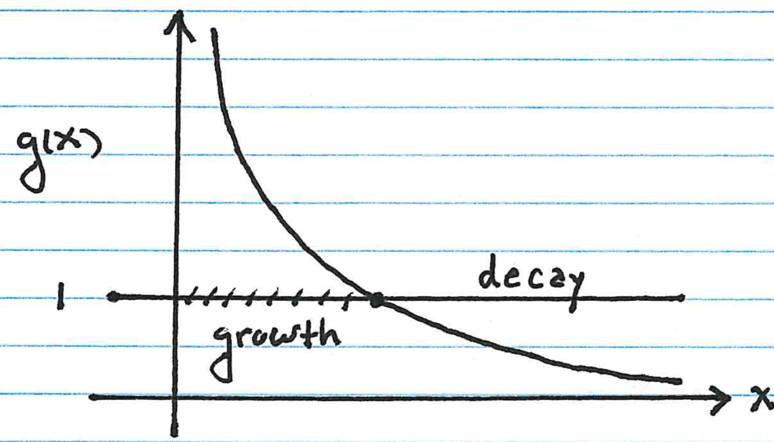
Is a graphical method for finding orbits.

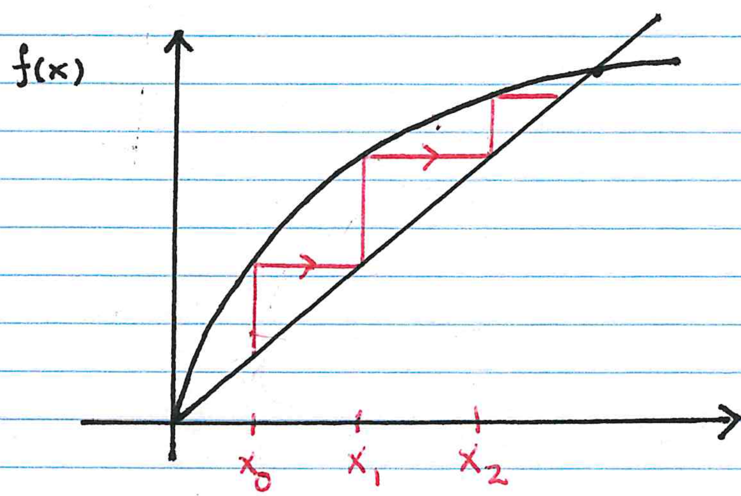




EXAMPLE Varley, Gradwell, Hassel (1975)

$$\boxed{g(x) = \lambda x^{-b} \qquad 0<b<1}$$

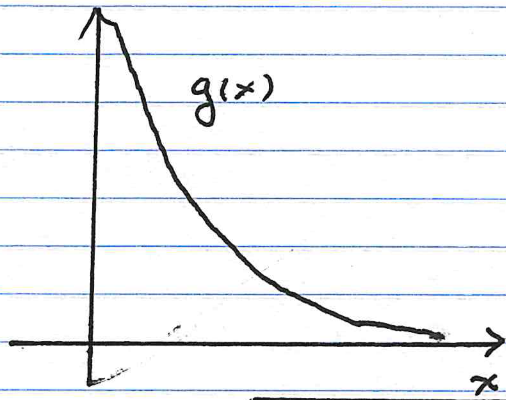


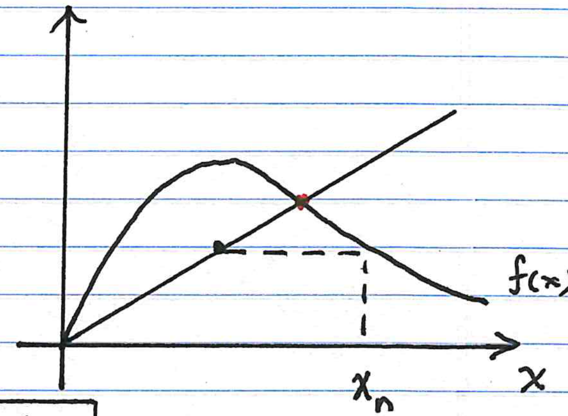


$f(x) = x\,g(x)$

$f(x) = \lambda x^{1-b}$

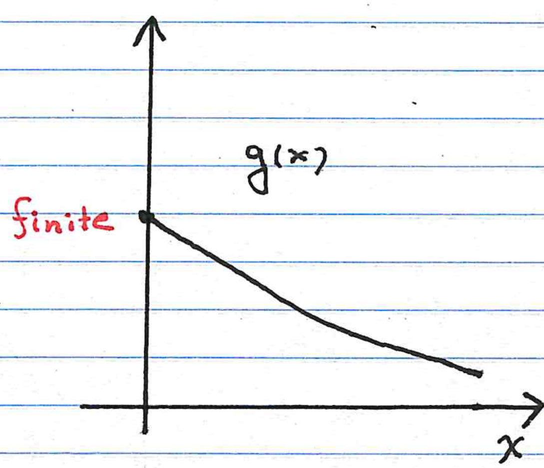
Ex May 1975

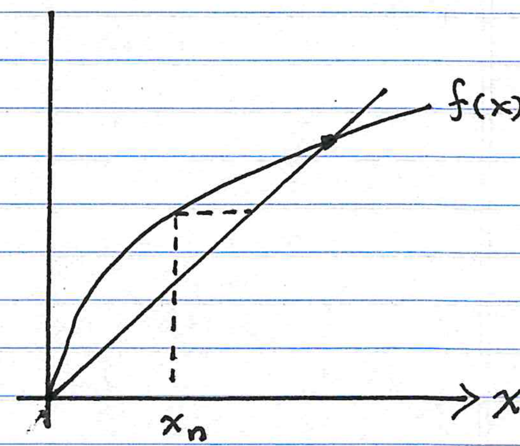




$$\boxed{g(x) = e^{r(1 - x/K)}}$$

Ex Hassel 1973





$$g(x) = \lambda (1 + ax)^{-b}$$

Fixed points and Linear stability

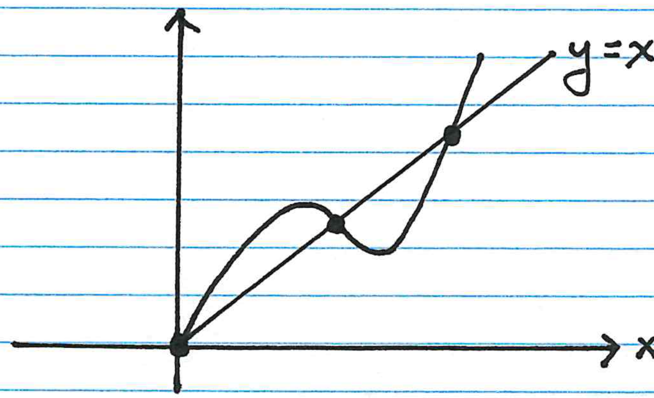
(1) $$x_{n+1} = f(x_n)$$

A fixed point of (1) is a value $\bar{x}$ such that

(2) $$\bar{x} = f(\bar{x})$$

For these values $x_n = \bar{x}$ for all n if $x_0 = \bar{x}$

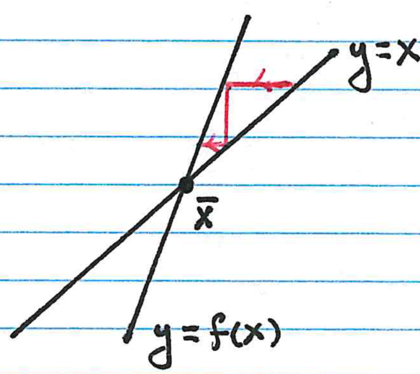
EX



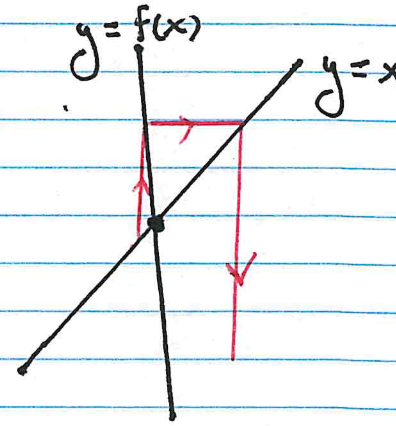
Has 3 fixed points

Linear stability

A fixed point $\bar{x}$ is linearly stable if $x_n \to \bar{x}$ for all x_0 sufficiently close to $\bar{x}$



$\bar{x}$ lin. stable



$\bar{x}$ unstable

Linear Stability

Let $\bar{x}$ be a fixed point of

(1) $\qquad x_{n+1} = f(x_n)$

Then

$$|f'(\bar{x})| < 1 \quad \Rightarrow \quad \bar{x} \text{ stable}$$

$$|f'(\bar{x})| > 1 \quad \Rightarrow \quad \bar{x} \text{ unstable}$$

Proof Sketch using Talyor series. Define

(2) $\qquad x_n = \bar{x} + \Delta x_n$

$|\Delta x_n|$ = distance from $\bar{x}$

Substitute (2) into (1)

$$x_{n+1} = f(x_n)$$

$$\bar{x} + \Delta x_{n+1} = f(\bar{x} + \Delta x_n)$$

(smaller)

$f(\bar{x}) = \bar{x}$ $\qquad \cancel{\bar{x}} + \Delta x_{n+1} = \cancel{f(\bar{x})} + f'(\bar{x})\Delta x_n + \frac{1}{2} f''(\bar{x}) \Delta x_n^2 + \cdots$

$$\boxed{\Delta x_{n+1} \cong f'(\bar{x}) \Delta x_n}$$

Looks like $\Delta x_{n+1} = \lambda \Delta x_n$ which decays only if $|\lambda| < 1$

$$|f'(\bar{x})| < 1 \quad \Rightarrow \quad \Delta x_n \to 0$$

$$|f'(\bar{x})| > 1 \quad \Rightarrow \quad \Delta x_n \to \infty$$

EXAMPLE

Find any positive fixed point(s) and stability of the difference equation (map)

$$x_{n+1} = x_n \ln x_n^2$$

$$x_{n+1} = f(x_n)$$

Here $f(x) = x \ln x^2$ and $\bar{x}$ is a fixed point if

$$\bar{x} = \bar{x} \ln \bar{x}^2$$

$$1 = \ln \bar{x}^2$$

$$e = \bar{x}^2$$

$$\boxed{e^{1/2} = \bar{x}}$$

To find the stability we need to compute $f'(\bar{x})$

$$f'(x) = \ln x^2 + x \cdot \left(\frac{1}{x^2} \cdot 2x\right)$$

$$f'(x) = \ln x^2 + 2$$

Hence

$$f'(\bar{x}) = \ln \bar{x}^2 + 2 = \ln e + 2 = 3$$

Since $|f'(\bar{x})| = 3 > 1$ we conclude $\bar{x} = e^{1/2}$ unstable

Logistic Model Analysis

(1) $$x_{n+1} = f(x_n) = r x_n (1 - x_n)$$

Solution behavior

$0 < r < 1$	extinction
$1 < r < 3$	stable fixed point
$3 < r < 1+\sqrt{6}$	period 2 solutions

To prove requires considerable work

Fixed Point existence and stability

(2) $$\bar{x} = f(\bar{x})$$

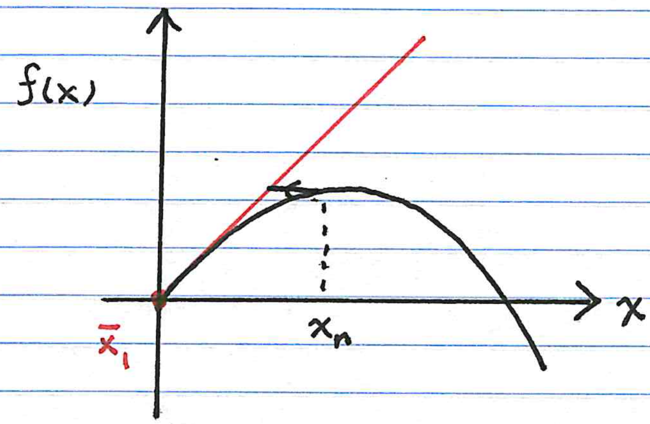
$$\bar{x} = r\bar{x}(1-\bar{x})$$

Solving (2) yields two fixed points

$$\bar{x}_1 = 0$$

$$\bar{x}_2 = 1 - \frac{1}{r} > 0 \qquad \text{(only if } r>1\text{)}$$

For $\bar{x}_1$ we use a cobweb argument.



"Global Stability"

Graph of $f = rx(1-x)$ for $r \in (0,1)$

Cobweb implies

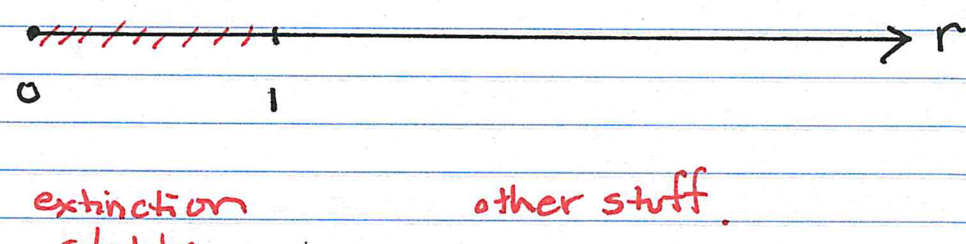
$$x_n \to 0$$

Can also check linear stability of $\bar{x}_1 = 0$

$$|f'(\bar{x}_1)| < 1$$

Explicitly this condition can be shown to be

$$|r| < 1$$



Likewise we can consider other fixed point

$$\bar{x}_2 = 1 - \frac{1}{r} > 0 \qquad (r > 1)$$

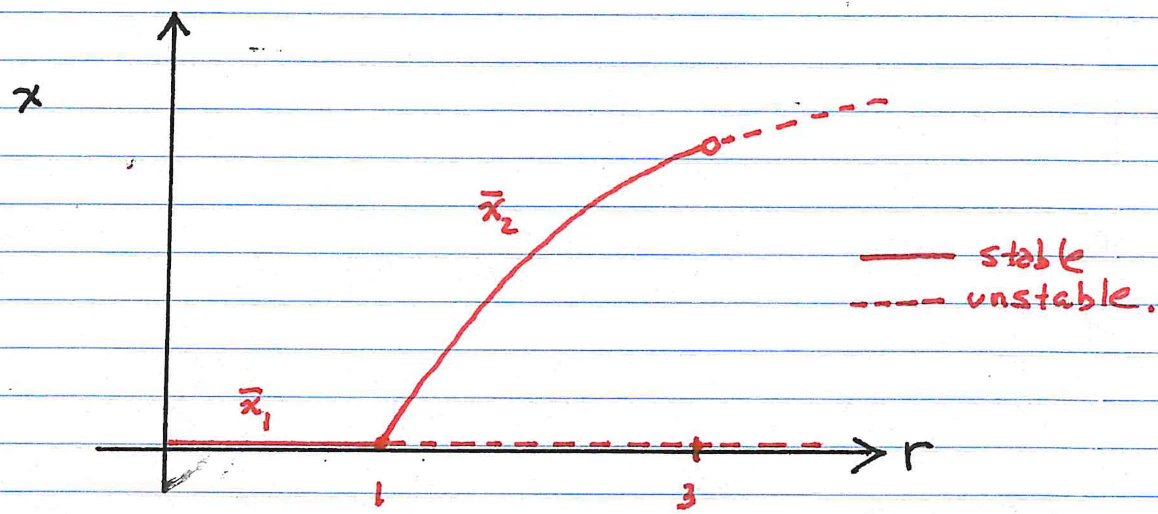
For this fixed point we have stability if

$$|f'(\bar{x}_2)| < 1$$

$$|2 - r| < 1$$

$$r \in (1, 3)$$

Bifurcation diagram $\bar{x}_1 = 0$, $\bar{x}_2 = 1 - \frac{1}{r}$



Note that fixed point $\bar{x}_2$ unstable for all $r > 3$

Period Doubling

(1) $$x_{n+1} = f(x_n)$$

Consider fixed points of the second iterate map. That is, consider roots x of

(2) $$x = f(f(x))$$

The orbit associated with x is

(3) $$\{ x, f(x), x, f(x), x, f(x), \ldots \}$$ period 2

This will be period 2 so long as $x \neq f(x)$

The latter corresponds to x not being a fixed point. Recall $\bar{x} = f(\bar{x})$ for a fixed point in which case

$$\bar{x} = f(f(\bar{x})) = f(\bar{x}) \qquad \checkmark$$

whence (3) would be

$$\{ \bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x}, \ldots \}$$ period 1

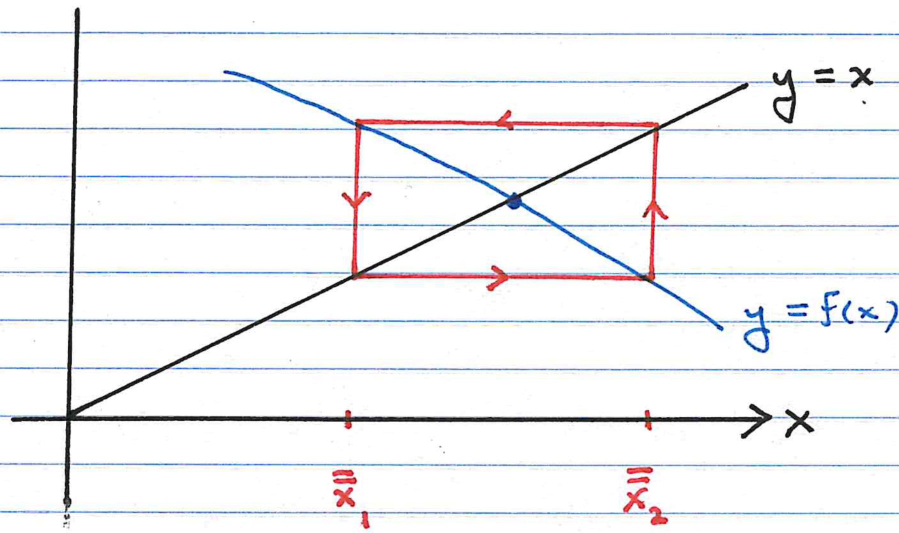
As in the text we shall use double overbar to denote true period two x values:

$$\bar{\bar{x}}_{1,2} = f(f(\bar{\bar{x}}_{1,2}))$$

So the orbit in (3) is

$$\{ \bar{\bar{x}}_1, \bar{\bar{x}}_2, \bar{\bar{x}}_1, \bar{\bar{x}}_2, \bar{\bar{x}}_1, \bar{\bar{x}}_2, \ldots \}$$

A typical period two cobweb looks like



<u>Stability</u> $\qquad g(x) \equiv f(f(x))$

A period two cycle is stable only if the fixed point of $g(x)$ is:

(5) $\qquad |g'(\bar{\bar{x}})| < 1 \qquad$ stable

Since

$$g'(x) = f'(f(x))\, f'(x)$$

and $\bar{\bar{x}}_2 = f(\bar{\bar{x}}_1)$ the stability condition (5) is:

$$\boxed{\left| f'(\bar{\bar{x}}_1)\, f'(\bar{\bar{x}}_2) \right| < 1}$$

stable period 2

Logistic Difference Eqn (Period 2)

$$x_{n+1} = f(x_n) \qquad f(x) = rx(1-x)$$

We already know two fixed points

(1) $$\bar{x}_1 = 0 \qquad \bar{x}_2 = 1 - \frac{1}{r}$$

To find period two orbits we need to find all the real roots of

$$x = f(f(x))$$

$$x = r\, f(x)\,(1 - f(x))$$

$$x = r^2 x(1-x)(1 - rx(1-x))$$

In summary we need the roots of the quartic

(2) $$\boxed{P(x) = r^2 x(1-x)(1-rx(1-x)) - x}$$

All is not lost. Two roots of (2) must be the fixed points $\bar{x}_k$. Long division yields (not shown)

quadratic

$$P(x) = x(x-\bar{x}_2)\,Q(x)$$

Roots of the quadratic $Q(x)$ can be found

(3) $$\boxed{\bar{\bar{x}}_{\pm} = \frac{r+1 \pm \sqrt{r^2-2r-3}}{2r}}$$

The condition that $\bar{\bar{x}}$ is <u>stable</u> is

$$(4) \qquad \boxed{\left| f'(\bar{\bar{x}}_+) f'(\bar{\bar{x}}_-) \right| < 1}$$

After a huge amount of calculations eqn (4) simplifies to:

$$(5) \qquad |4 + 2r - r^2| < 1$$

We find a range of r for stability by setting the quadratic in (5) equal to ± 1

$$4 + 2r - r^2 = +1 \quad \Rightarrow \quad r = 3$$

$$4 + 2r - r^2 = -1 \quad \Rightarrow \quad r = 1 + \sqrt{6}$$

Conclude period 2 orbit stable only if

$$3 < r < 1 + \sqrt{6} \simeq 3.45$$

Logistic model Bifurcation diagram

Plot fixed points $\bar{x}_1, \bar{x}_2$ and period 2 $\bar{\bar{x}}_{\pm}$.

By convention solid line means stable, dashed line means unstable