Planar Differential Systems

$$(1) \qquad \boxed{\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}}$$

Some systems like (1) can be solved explicitly. For instance the linear system

$$(2) \qquad x' = -y$$

$$(3) \qquad y' = x$$

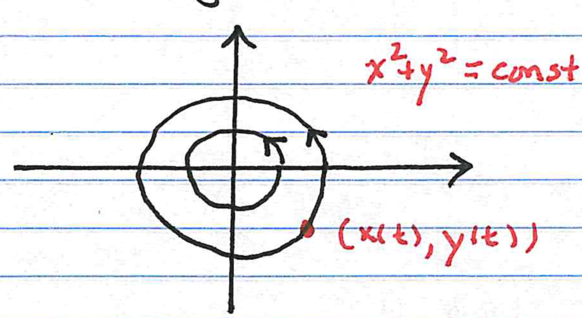
can be shown to have the general soln (Math 274)

$$(4) \qquad \vec{x}(t) = c_1 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

Such explicit formulae are not always informative but do illustrate solutions (or trajectories) are curves in the (x,y)-plane. We often resort to "qualitate" techniques to get a better idea of what these trajectories look like. For (2)-(3) note:

$$\frac{d}{dt}(x^2+y^2) = 2xx' + 2yy' = -2xy + 2xy = 0$$

says its trajectories are circles

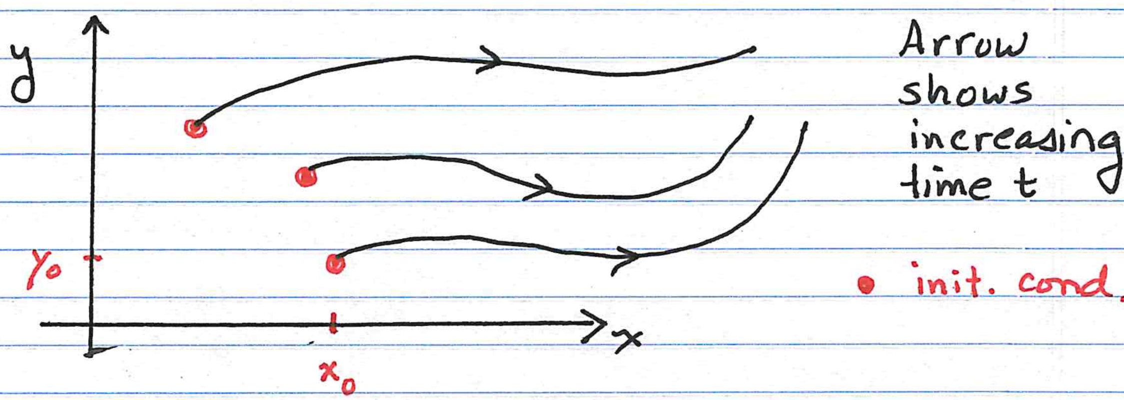


concentric circles for diff init cond.

Again lets recall our general planar problem

$$(1) \qquad \boxed{\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}}$$

Each initial condition pair generates its own unique trajectory (or solution). Potentially it might look like



If we pose (1) as a vector equation we can say more

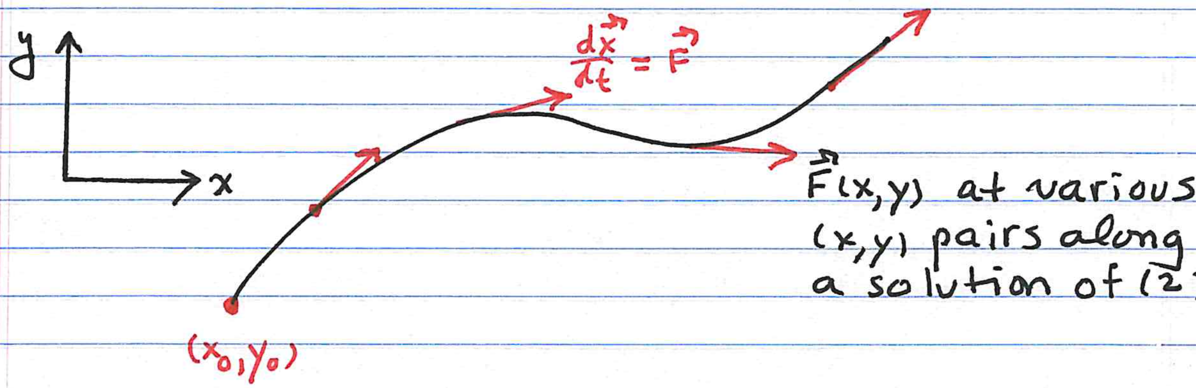
$$(2) \qquad \frac{d\vec{x}}{dt} = \vec{F}(x,y)$$

is equvalent to

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \vec{F}(x,y)$$

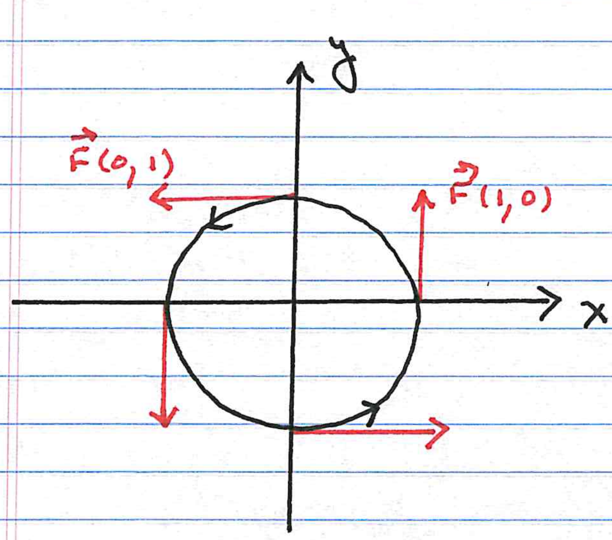
In (2), $\vec{F}(x,y)$ is a known vector field which can be drawn independent of knowing the solution $\vec{x}(t)$.

More importantly (2) states that solutions $\vec{x}(t)$ are everywhere tangent to the vector field $\vec{F}(x,y)$



EXAMPLE

$$x' = -y \qquad \vec{F}(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}$$
$$y' = +x$$



(x, y)	$\vec{F}(x,y)$	
$(1, 0)$	$(0, 1)$	$\uparrow$
$(0, 1)$	$(-1, 0)$	$\leftarrow$
$(-1, 0)$	$(0, -1)$	$\downarrow$
$(0, -1)$	$(1, 0)$	$\rightarrow$

$\boxed{\vec{F}(x,y) \text{ is the direction field for the system}}$

Equilibria and Nullclines

Recall the system

(1) $\quad x' = f(x,y)$

(2) $\quad y' = g(x,y)$

Drawing $\vec{F}(x,y)$ can result in a cluttered mess in the (x,y) phase plane. Typically one uses only a limited selection of $\vec{F}(x,y)$ to infer the direction (or flow) of the solution.

For example, where is the flow horizontal? where is it vertical?

Consider $f(x,y)=0$. This is a curve(s) called the x-nullcline. For (x,y) on this curve

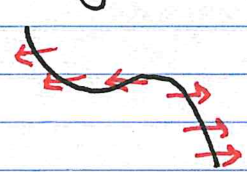
$$\vec{F} = \begin{pmatrix} 0 \\ g(x,y) \end{pmatrix}$$

$f(x,y)=0$

and $\vec{F}$ is vertical.

Consider $g(x,y)=0$. This is a curve(s) called the y-nullcline. For (x,y) on this curve

$$\vec{F} = \begin{pmatrix} f(x,y) \\ 0 \end{pmatrix}$$



$g(x,y)$

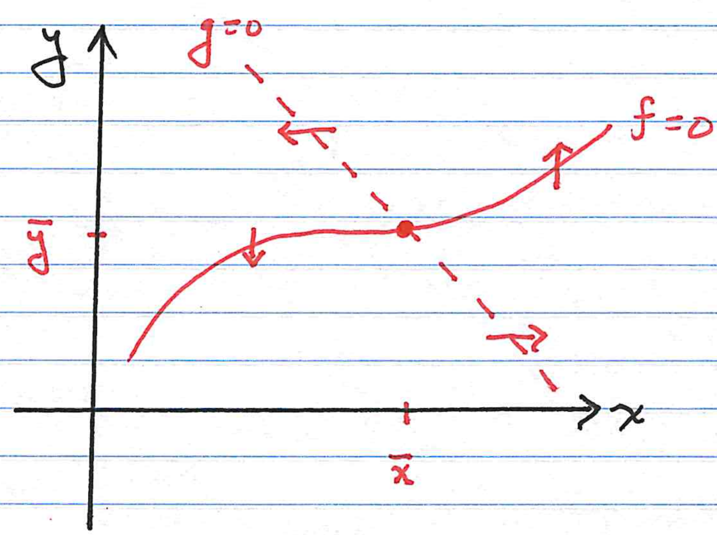
and $\vec{F}$ is horizontal

Lastly consider equilibria. At equilibria both f and g vanish.

We shall use overbar notation to denote these special points

$$(1) \qquad \boxed{\begin{aligned} f(\bar{x}, \bar{y}) &= 0 \\ g(\bar{x}, \bar{y}) &= 0 \end{aligned}}$$

Equilibria occur at the intersection of the x and y nullclines



To find equilibria you must find all solutions to the algebraic problem (1) above. Generally these are two nonlinear equations and may have no soln, one solution or many solutions.

Equilibria summary

Found by solving the coupled algebraic sys:

$$f(\bar{x}, \bar{y}) = 0$$

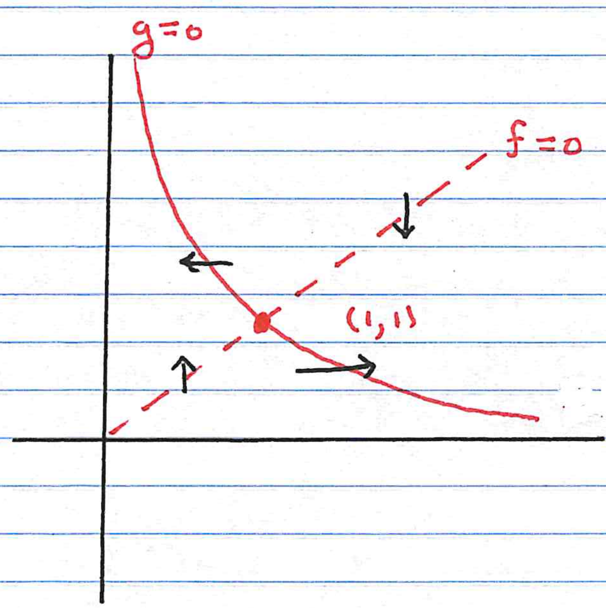
$$g(\bar{x}, \bar{y}) = 0$$

They occur where nullclines intersect <u>and</u> if the initial condition is $(\bar{x}, \bar{y})$ the solution is $x(t) = \bar{x}$, $y(t) = \bar{y}$ for all time t.

$$\boxed{\begin{array}{ll} x' = f(x,y), & x(0) = \bar{x} \\ y' = g(x,y), & y(0) = \bar{y} \end{array}} \Rightarrow \begin{array}{l} x(t) = \bar{x} \\ y(t) = \bar{y} \end{array}$$

EXAMPLE

$$\boxed{\begin{array}{l} x' = x - y \\ y' = 1 - xy \end{array}}$$



<u>x-nullcline</u>

$f = 0 \Rightarrow y = x$

<u>y-nullcline</u>

$g = 0 \Rightarrow y = \frac{1}{x}$

<u>Equilibria</u>

$$x - y = 0$$
$$1 - xy = 0$$

yields $x^2 = 1$ or $x = \pm 1$ hence

$(1,1)$ $(-1,-1)$

Linear System Review - Planar

(1) $$\frac{d\vec{x}}{dt} = A\vec{x} \qquad \vec{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Eigenvalues and eigenvectors of the matrix A determine the general solution of (1)
Letting

(2) $$\vec{x}(t) = e^{\lambda t}\vec{\xi} \qquad \vec{\xi} \text{ const. vector}$$

we find (2) is a solution of (1) only if

(3) $$A\vec{\xi} = \lambda\vec{\xi}$$

Moreover, for $\vec{\xi} \neq 0$ we need $(A-\lambda I)$ not invertible, or, its determinant must vanish

(4) $$P(\lambda) = \det(A-\lambda I) \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For $A \in \mathbb{R}^{2\times 2}$ this characteristic polynomial is a quadratic with two roots called eigenvalues. The vectors $\vec{\xi}$ in (3) are not unique and are called eigenvectors

Summary

$$\vec{x}(t) = e^{\lambda t}\vec{\xi}$$

$$\det(A-\lambda I) = 0$$

$$A\vec{\xi} = \lambda\vec{\xi}$$

Real Distinct eigenvalues $\lambda_1 \neq \lambda_2 \in \mathbb{R}$

The general solution of (1) is

$$\vec{x} = c_1 e^{\lambda_1 t} \underset{\uparrow}{\vec{\xi}_1} + c_2 e^{\lambda_2 t} \underset{\uparrow}{\vec{\xi}_2}$$

eigenvectors for λ_1, λ_2

EXAMPLE

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} \vec{x} = A\vec{x}$$

$$P(\lambda) = \det(A - \lambda I) = (\lambda - 3)(\lambda - 4)$$

after some calculations.

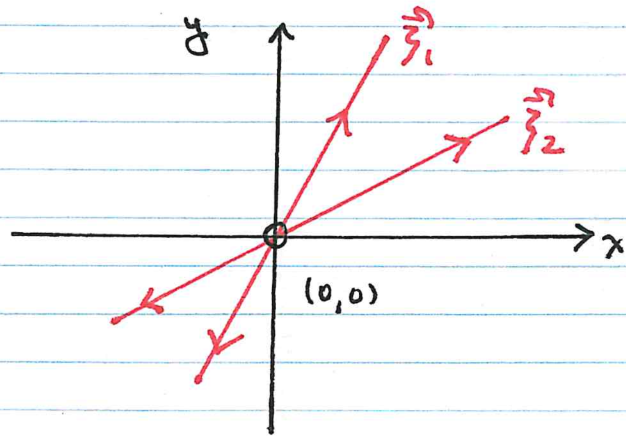
$\lambda_1 = 3 \qquad (A - \lambda_1 I)\vec{\xi} = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} \vec{\xi} = \vec{0} \qquad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 4 \qquad (A - \lambda_2 I)\vec{\xi} = \begin{bmatrix} 2 & -3 \\ * & * \end{bmatrix} \vec{\xi} = \vec{0} \qquad \vec{\xi}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

General solution

$$\vec{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

grows.



Unstable Node

EXAMPLE (Saddle) $\lambda_2 < 0 < \lambda_1$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} = A\vec{x}$$

Here the characteristic polynomial is

$$P(\lambda) = \lambda^2 - 1 \qquad \lambda = \pm 1$$

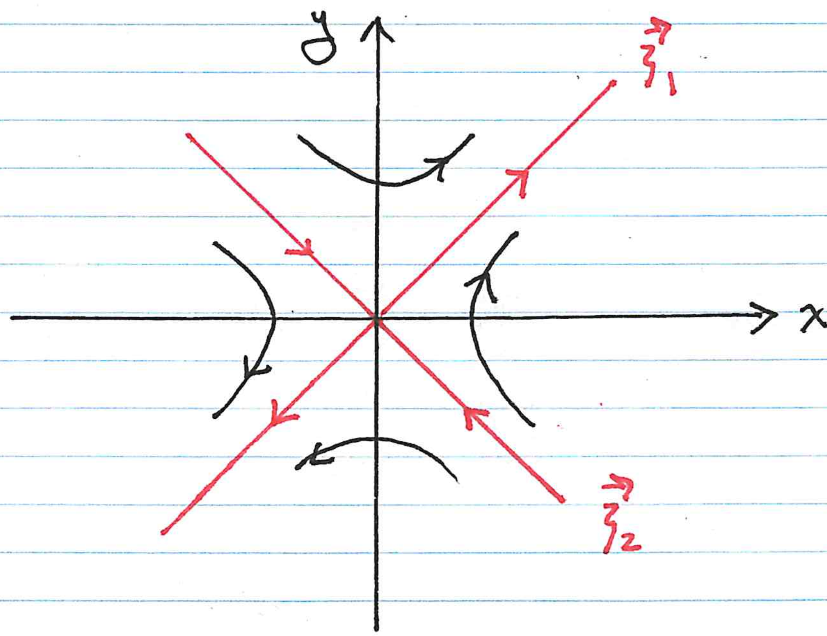
It is easily verified we have eigenvectors:

$$\lambda_1 = +1 \qquad (A - \lambda_1 I)\vec{\xi} = \begin{bmatrix} -1 & 1 \\ * & * \end{bmatrix}\vec{\xi} \qquad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \qquad (A - \lambda_2 I)\vec{\xi} = \begin{bmatrix} 1 & 1 \\ * & * \end{bmatrix}\vec{\xi} \qquad \vec{\xi}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

General Solution

$$\vec{x}(t) = \overbrace{c_1 e^{\lambda_1 t}}^{\text{grows}} \vec{\xi}_1 + \overbrace{c_2 e^{\lambda_2 t}}^{\text{decays}} \vec{\xi}_2$$



Complex eigenvalues $\lambda = \alpha + i\beta \in \mathbb{C}$

Eigenvalues and eigenvectors are both complex.
The complex solution is

$$\vec{x}(t) = e^{\lambda t}\,\vec{\xi} = \vec{x}_r(t) + i\,\vec{x}_i(t)$$

Each of the real and imaginary parts of $\vec{x}$ are independent solutions to

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

After computing $\vec{\xi} = \vec{a} + i\vec{b}$ and expanding

$$\vec{x} = e^{(\alpha + i\beta)t}\,(\vec{a} + i\vec{b})$$

and using $e^{i\beta t} = \cos\beta t + i\sin\beta t$, two real independent solutions are

$$\vec{x}_1(t) = e^{\alpha t}(\cos\beta t\,\vec{a} - \sin\beta t\,\vec{b})$$

$$\vec{x}_2(t) = e^{\alpha t}(\sin\beta t\,\vec{a} + \cos\beta t\,\vec{b})$$

Note even though solns always oscillate

$\alpha > 0$	$\vec{x}_K$ grow
$\alpha < 0$	$\vec{x}_K \to 0$
$\alpha = 0$	$\vec{x}_K$ periodic

Here β is the frequency.

EXAMPLE Simple Purely imaginary case $\lambda = i\beta$

$$\vec{x}' = A\vec{x} = \begin{bmatrix} 0 & -8 \\ 2 & 0 \end{bmatrix} \vec{x}$$

Characteristic Polynomial

$$P(\lambda) = \lambda^2 + 16 \qquad \lambda = 4i$$

Eigenvector

$$(A - \lambda I) = \begin{bmatrix} -4i & -8 \\ 2 & -4i \end{bmatrix} \qquad \vec{\xi} = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

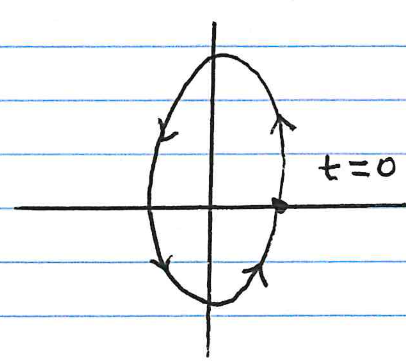
General solution

$$\vec{x}(t) = c_1 \begin{pmatrix} -2\sin 4t \\ \cos 4t \end{pmatrix} + c_2 \underbrace{\begin{pmatrix} 2\cos 4t \\ \sin(4t) \end{pmatrix}}_{\vec{x}_2(t)}$$

To see what these trajectories, look at $\vec{x}_2(t)$

$$\vec{x}_2 = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2\cos 4t \\ \sin 4t \end{pmatrix} \Rightarrow \left(\frac{x}{2}\right)^2 + y^2 = 1$$

is on ellipse



EXAMPLE Complex = Spiral

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x} = A\vec{x}$$

Characteristic Polynomial

$$P(\lambda) = \lambda^2 - 2\lambda + 2 = 0 \quad \Rightarrow \quad \lambda = \overset{\alpha + i\beta}{1 + i}$$

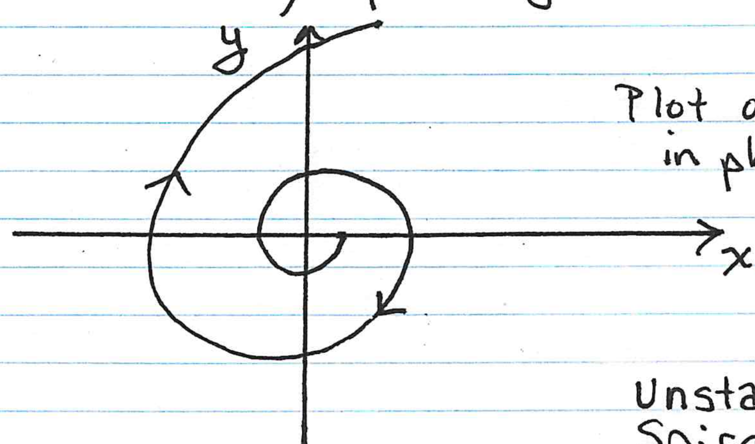
with associated eigenvector

$$\vec{\xi} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{a} + i\vec{b}$$

Ultimately $\vec{x} = e^t(\cos t + i \sin t)(\vec{a} + i\vec{b})$ yields the general solution

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) = c_1 \underbrace{\begin{pmatrix} e^t \cos t \\ -e^t \sin t \end{pmatrix}}_{\vec{x}_1(t)} + c_2 \underbrace{\begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix}}_{\vec{x}_2(t)}$$

Notice $\|x_1(t)\| = e^t$ so $\vec{x}_1(t) \to \infty$ as $t \to \infty$ but it does so by spiraling outward. Same with $\vec{x}_2$



Plot of $\vec{x}_1(t)$ in phase plane.

Unstable Spiral

EXAMPLE Purely Imaginary $\lambda = i\beta$

$$\vec{x}' = A\vec{x} \qquad A = \begin{bmatrix} -2 & -4 \\ 10 & 2 \end{bmatrix}$$

Characteristic Polynomial (after calculations)

$$P(\lambda) = \lambda^2 + 36 = 0 \qquad \lambda = 6i \qquad (\alpha = 0)$$

Eigenvector

$$(A - \lambda I) = \begin{bmatrix} -2-6i & -4 \\ * & * \end{bmatrix} \qquad \vec{\xi} = \begin{pmatrix} 2 \\ -1-3i \end{pmatrix}$$

General solution

$$\vec{x}(t) = c_1 \underbrace{\begin{pmatrix} 2\cos 6t \\ -\cos 6t + 3\sin 6t \end{pmatrix}}_{\vec{x}_1(t)} + c_2 \begin{pmatrix} 2\sin 6t \\ -\sin 6t - 3\cos 6t \end{pmatrix}$$

Are ellipses with different axes. From $\vec{x}_1(t)$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{x}{6} + \frac{y}{3}\right)^2 = 1$$

$$\cos^2 + \sin^2 = 1$$

Has axes:

$$x = 0 \qquad x + 2y = 0$$

Linearization about equilibria

A useful tool in the study of ODE models is the process of linearization. Typically one wants to know if some given equilibria is stable or not.

Let $(\bar{x}, \bar{y})$ be an equilibria of

$$(1) \qquad \dot{x} = f(x,y)$$

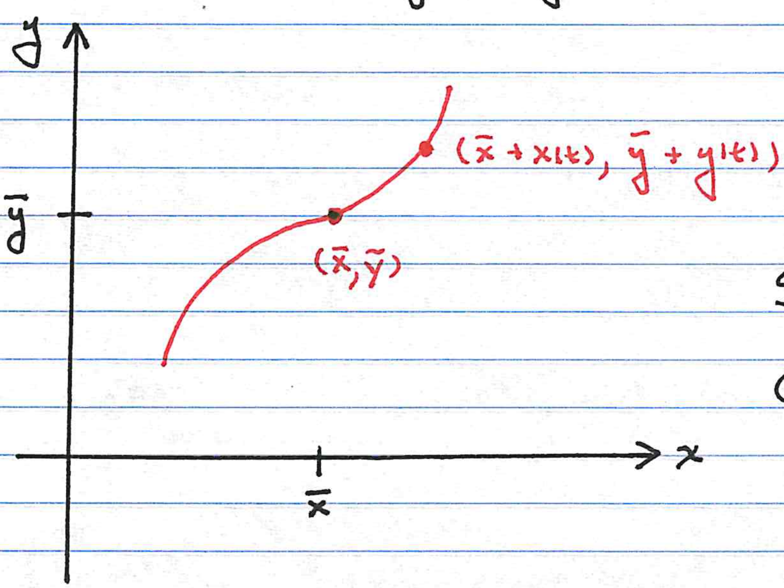
$$(2) \qquad \dot{y} = g(x,y)$$

Then

$$(3) \qquad f(\bar{x}, \bar{y}) = 0$$
$$(4) \qquad g(\bar{x}, \bar{y}) = 0$$

Let $(X(t), Y(t))$ be solution trajectories near the equilibria

$$(5) \qquad X(t) = \bar{x} + x(t)$$
$$(6) \qquad Y(t) = \bar{y} + y(t)$$



Small x, y

Close to $(\bar{x}, \bar{y})$!

Start with

(1) $$\mathbb{X}' = f(\mathbb{X}, \mathbb{Y})$$

(2) $$\mathbb{Y}' = g(\mathbb{X}, \mathbb{Y})$$

where

$$\mathbb{X}(t) = \bar{x} + x(t)$$
$$\mathbb{Y}(t) = \bar{y} + y(t)$$

Then since $(\bar{x}, \bar{y})$ are constant

(3) $$x' = f(\bar{x}+x, \bar{y}+y)$$

(4) $$y' = g(\bar{x}+x, \bar{y}+y)$$

A Taylor series (linear approx) of (3)-(4) is:

$$x' = \cancel{f(\bar{x},\bar{y})}^{0} + f_x(\bar{x},\bar{y})\,x + f_y(\bar{x},\bar{y})\,y + \text{smaller}$$
$$y' = \cancel{g(\bar{x},\bar{y})}^{0} + g_x(\bar{x},\bar{y})\,x + g_y(\bar{x},\bar{y})\,y + \text{smaller}$$

Noting first terms vanish since $(\bar{x},\bar{y})$ an equilibria and neglecting the smaller terms yields the linearized system:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{bmatrix} f_x(\bar{x},\bar{y}) & f_y(\bar{x},\bar{y}) \\ g_x(\bar{x},\bar{y}) & g_y(\bar{x},\bar{y}) \end{bmatrix}}_{\text{called the Jacobian matrix } D\vec{F}(\bar{x},\bar{y})} \begin{pmatrix} x \\ y \end{pmatrix}$$

Summary of Linearization

$$\begin{aligned} X' &= f(X, Y) \\ Y' &= g(X, Y) \end{aligned}$$ original system

Find equilibria

$$\begin{aligned} 0 &= f(\bar{x}, \bar{y}) \\ 0 &= g(\bar{x}, \bar{y}) \end{aligned}$$ equilibria eqns

There may be more than one equilibria satisfying the two eqns above.

$$\begin{aligned} X(t) &= \bar{x} + x(t) \\ Y(t) &= \bar{y} + y(t) \end{aligned}$$

Linearized system in vector form $\vec{x} = (x, y)$

(1) $$\frac{d\vec{x}}{dt} = A\vec{x}$$ $A = D\vec{F}(\bar{x}, \bar{y})$ Jacobian

where the Jacobian

$$A = D\vec{F}(\bar{x}, \bar{y}) = \begin{bmatrix} f_x(\bar{x}, \bar{y}) & f_y(\bar{x}, \bar{y}) \\ g_x(\bar{x}, \bar{y}) & g_y(\bar{x}, \bar{y}) \end{bmatrix}$$

Given how $\vec{x}(t)$ is defined, the system (1) describes the flow/solution only near the equilibria.

Application to linearized system (example)

$$x' = f(x,y) = (x-1)y$$
$$y' = g(x,y) = xy - x$$

Can easily verify the system has two equilibria

$$P_0 = (0,0) \qquad P_1 = (1,1)$$

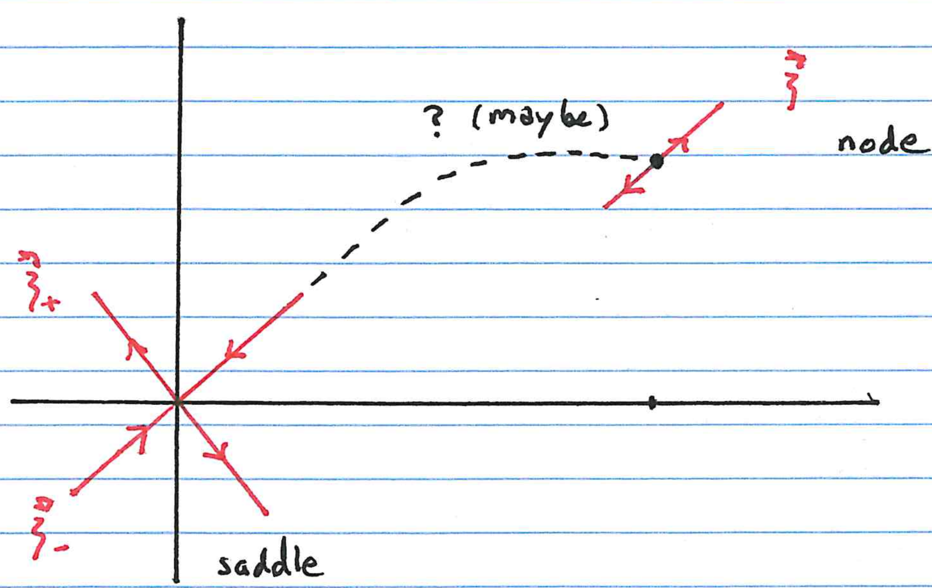
Jacobian at (x,y)

$$DF(x,y) = \begin{bmatrix} y & (x-1) \\ y-1 & x \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$

Jacobian at P_0, P_1 and e-vals:

$$A_0 = DF(P_0) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad \lambda_\pm = \pm 1 \qquad \text{saddle}$$

$$A_1 = DF(P_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \lambda = 1 \text{ repeated} \qquad \text{unstable node}$$



Classification of equilibria

$$(1) \qquad \vec{x}\,' = A\vec{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$$

After some calculations the characteristic polynomial is found

$$P(\lambda) = \det(A - \lambda I)$$

$$P(\lambda) = \lambda^2 - (a+d)\lambda + (ad-bc)$$

or in terms of the trace TrA and determinant of the matrix

$$(2) \qquad \boxed{P(\lambda) = \lambda^2 - \mathrm{Tr}A\,\lambda + \det A}$$

where

$$\mathrm{Tr}A = a+d$$

$$\det A = ad - bc$$

The stability of (1) will then be determined by the sign of the real part of the eigenvalues

$$(3) \qquad \boxed{\lambda_{\pm} = \frac{1}{2}\left(\mathrm{Tr}A \pm \sqrt{\mathrm{Tr}A^2 - 4\det A}\right)}$$

Again, whether $\lambda_\pm$ are real or complex the real part alone determines stability.
To a large extent the sign of $\mathrm{Tr}A^2 - 4\det A$ matters

Case: $\det A < 0$

$\lambda_- < 0 < \lambda_+$ unstable saddle

Case: $\det A > 0$, $\mathrm{Tr}A^2 - 4\det A > 0$

$\mathrm{Tr}A > 0$ $\lambda_+ > \lambda_- > 0$ unstable node

$\mathrm{Tr}A < 0$ $\lambda_- < \lambda_+ < 0$ stable node

Case: $\det A > 0$, $\mathrm{Tr}A^2 - 4\det A < 0$

$\lambda = \alpha \pm i\beta$ complex $\beta = \frac{1}{2}\sqrt{4\det A - \mathrm{Tr}A^2}$

$\mathrm{Tr}A > 0$ unstable spiral

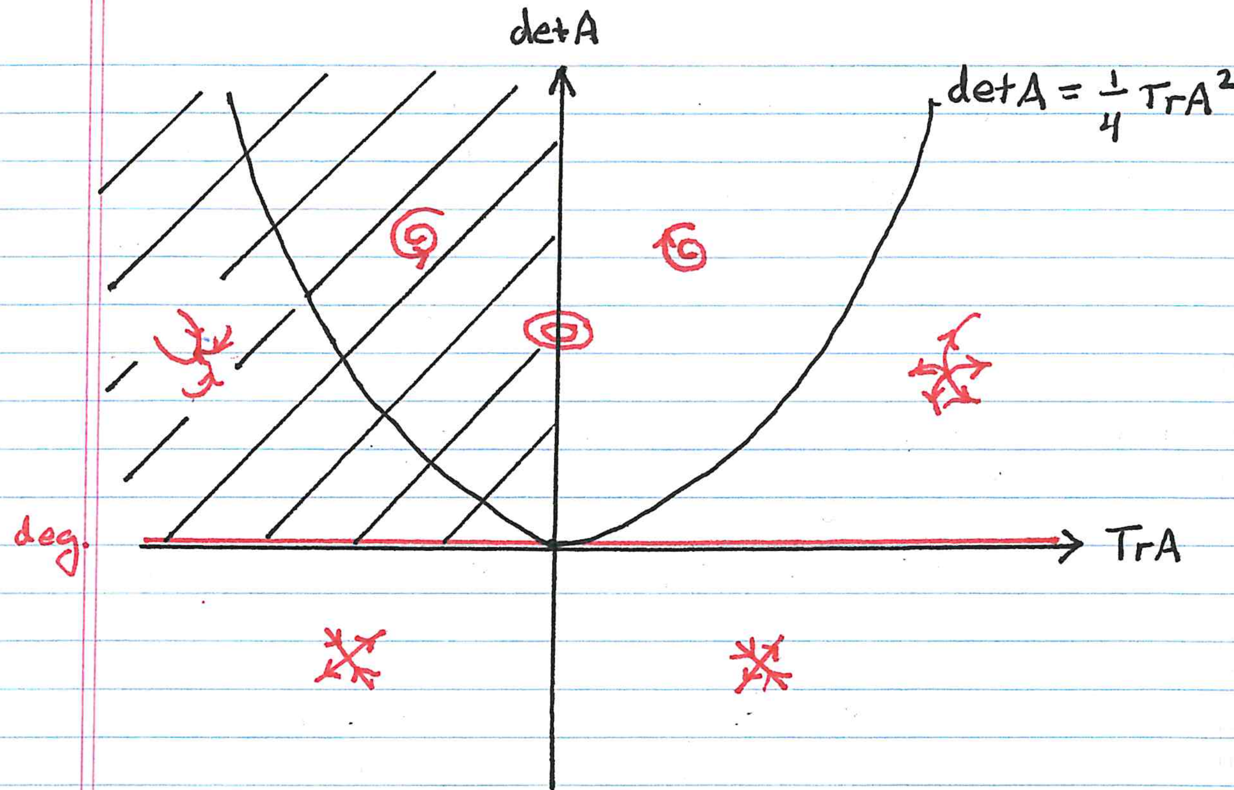
$\mathrm{Tr}A < 0$ stable spiral

Case: $\det A > 0$, $\mathrm{Tr}A = 0$

$\lambda = i\beta$ center

Case: $\det A = 0$

degenerate line of equilibria



$$\boxed{\mathrm{Tr}A < 0, \quad \det A > 0}$$

for stability. Otherwise $\vec{x} = 0$ is unstable

Applications to nonlinear systems $\vec{x}' = \vec{f}(x,y)$

If we get a linearized system

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad A = D\vec{F}(\bar{x}, \bar{y})$$

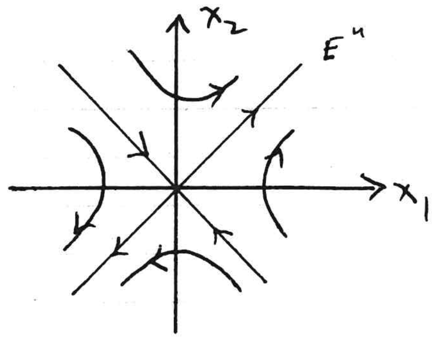
where one of the following is true

$$\mathrm{Tr}A = 0 \qquad \det A = 0$$

we can't conclude anything. Such points are "nonhyperbolic" and the linearized system doesn't necessarily well approximate the true system. We exclude all this theory.

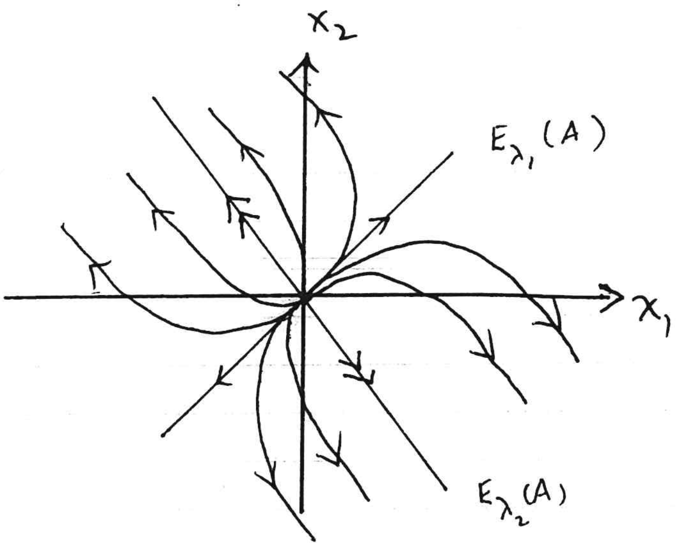
Classification Examples

EX **SADDLE** $\quad A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \qquad \lambda_{\pm} = \pm 2, \quad \underset{E^u}{\xi_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \quad \underset{E^s}{\xi_- = \begin{pmatrix} 1 \\ -3 \end{pmatrix}}$



EX **UNSTABLE NODE** $\quad A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \qquad \lambda_1 = 2 \qquad \lambda_2 = 4$

$$\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \xi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



$$x(t) = c_1 e^{2t} \vec{\xi}_1 + c_2 e^{4t} \vec{\xi}_2$$

$$x(t) \sim c_2 e^{4t} \vec{\xi}_2 \quad \text{as} \quad t \to \infty$$

Ex **Center** $A = \begin{bmatrix} 4 & -4 \\ 20 & -4 \end{bmatrix}$ $\lambda = \pm 2i$; $T = \frac{2\pi}{\omega} = \pi$ period.

$$z = \begin{pmatrix} 2 \\ 2-i \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$z = z_R + i z_I.$$

$$x(t) = c_1 u(t) + c_2 v(t)$$

$u(t) = \cos 2t \, z_R - \sin 2t \, z_I$ $u(0) = z_R$ u

$v(t) = \sin 2t \, z_R + \cos 2t \, z_I$ $v(0) = z_I$

Suppose I.C. chosen so that $x(0) = c_1 u(0) = x_0$ with $c_1 = 1$

$x(t) = u(t)$

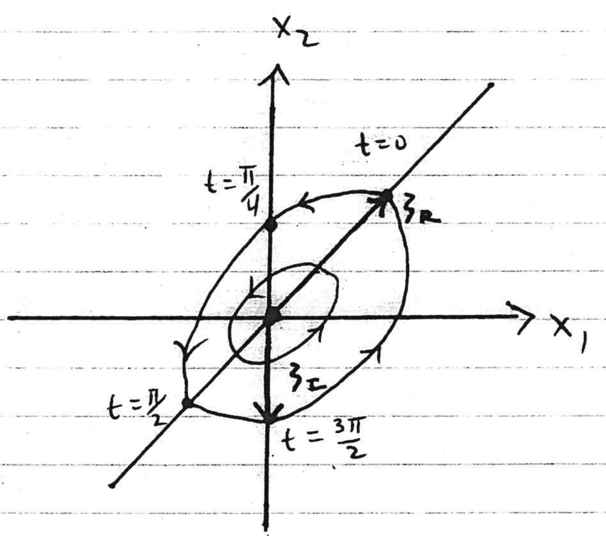
$u(0) = z_R$

$u(\pi/4) = -z_I$

$u(\frac{\pi}{2}) = -z_R$

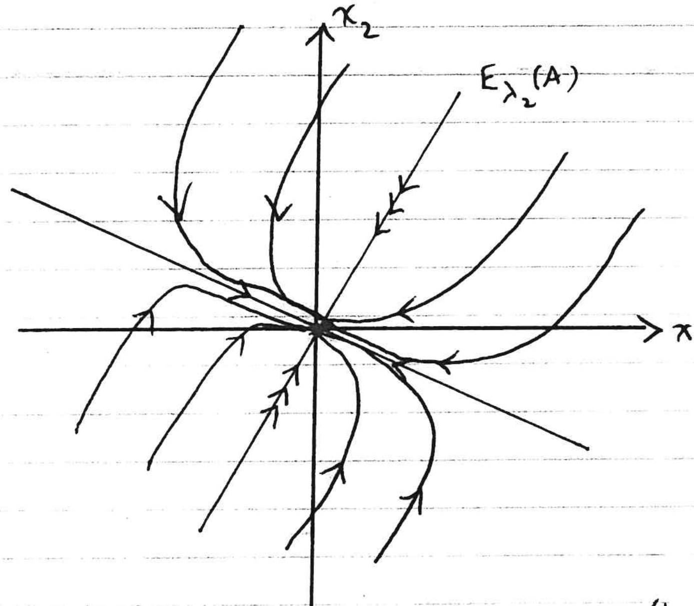
$u(\frac{3\pi}{4}) = +z_I$

$u(\pi) = z_R$



Neutral Stable.

EX **STABLE NODE** $A = \begin{bmatrix} -13 & -27 \\ -9 & -31 \end{bmatrix}$ $\lambda_1 = -4 \quad \vec{\xi}_1 = (-3, 1)^T$
$\lambda_2 = -40 \quad \vec{\xi}_2 = (1, 1)^T$



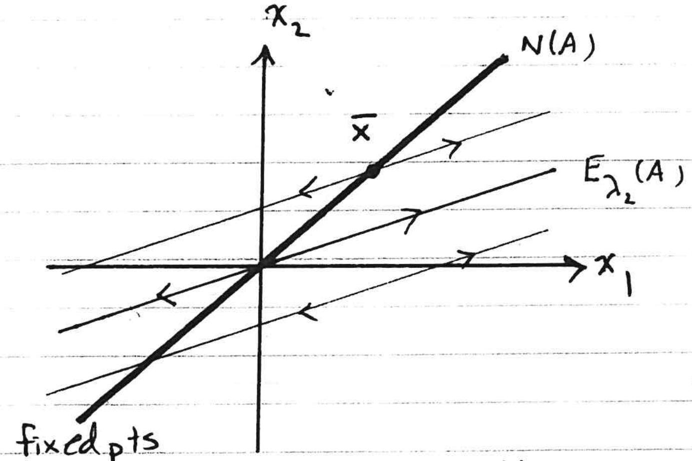
$x(t) = c_1 e^{-4t} \vec{\xi}_1 + c_2 e^{-40t} \vec{\xi}_2$

E

$E^s(0) = \mathbb{R}^2$
$E^u(0) = \{0\}$

" squashed onto $E_{\lambda_1}(A)$ "

EX **DEGENERATE (LINE OF FIXED)** $A = \begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix}$



$\lambda_1 = 0 \quad \vec{\xi}_1 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$

$\lambda_2 = 1 \quad \vec{\xi}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$x(t) = c_1 \vec{\xi}_1 + c_2 e^t \vec{\xi}_2$

Non isolated, Neutral stability.

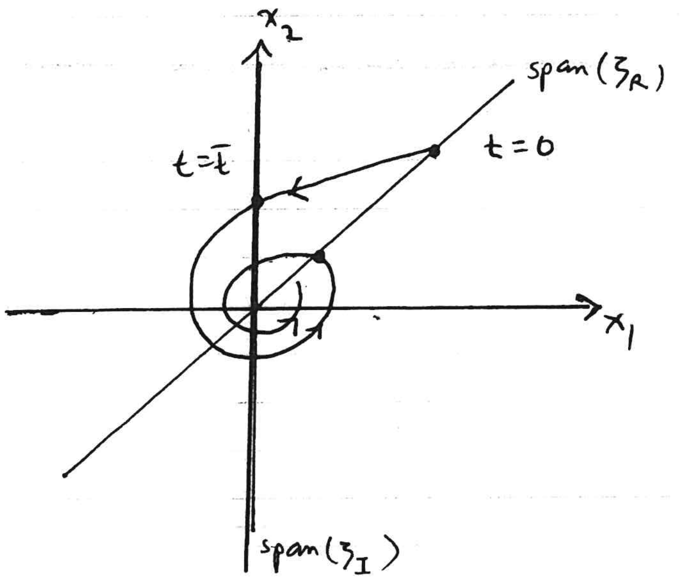
$E^s(\bar{x}) = \{\bar{x}\}$. However, for $\bar{x} = (\bar{x}_1, \bar{x}_2)$

$$E^u(\bar{x}) = \{(x_1, x_2) : \ x_2 - \bar{x}_2 = \tfrac{1}{2}(x_1 - \bar{x}_1)\}$$

is a straight line. $E^s(\bar{x}) \cup E^u(\bar{x}) \neq \mathbb{R}^2$ in this case

EX STABLE SPIRAL

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -2 \end{bmatrix}$$



$$\lambda_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2}\, i$$

$$\zeta = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + i \begin{pmatrix} 0 \\ -\sqrt{15} \end{pmatrix} = \vec{\zeta}_R + i\vec{\zeta}_i$$

$\omega = \frac{\sqrt{15}}{2}$; NOT PERIODIC.

Choose I.C. so that

$$x(t) = u(t) = e^{-\frac{1}{2}t}\left(\cos\omega t\, \vec{\zeta}_R - \sin\omega t\, \vec{\zeta}_I\right)$$

Then, when $t = \bar{t} = \frac{\pi}{2\omega}$, $x(\bar{t}) = -e^{-\frac{1}{2}\bar{t}}\, \vec{\zeta}_I$

<u>EXAMPLE</u>

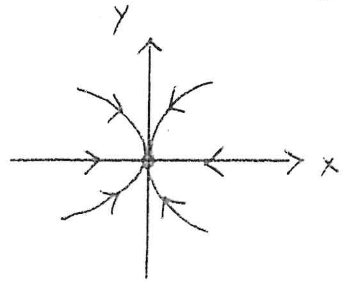
$$\dot{x} = ax \qquad \dot{y} = -y$$

$(\bar{x}, \bar{y}) = (0, 0)$ sole fx pt. if $a \neq 0$.

Solution for $x(0) = x_0$, $y(0) = y_0$ is

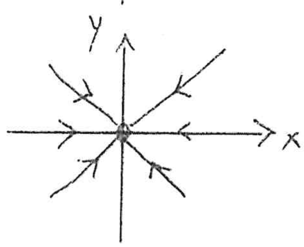
$$x(t) = x_0 e^{at} \qquad y(t) = y_0 e^{-t}$$

$a < -1$



isolated, a.s. NODE

$a = -1$

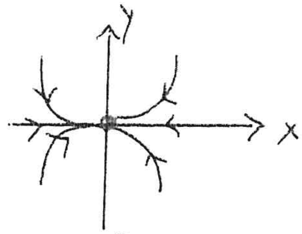


$\frac{y(t)}{x(t)} = K = \frac{y_0}{x_0}$, $\forall t \in \mathbb{R}$

so long as $x(t) \neq 0$.

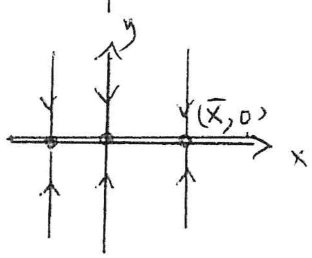
isolated, a.s. NODE (Star)

$a \in (-1, 0)$



isolated, a.s. NODE

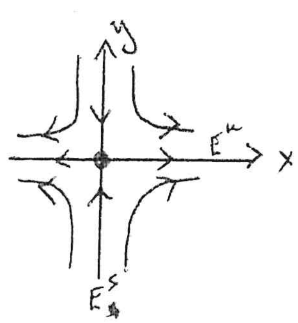
$a = 0$



non isolated. $y = 0$, $x \in \mathbb{R}$ line of fixed pts.

NOT ATTRACTING, IS LIAPUNOV $\Rightarrow$ Neutral.

$a > 0$



UNSTABLE. SADDLE

(Stable and unstable manifolds)