Chemostat Model Analysis

$$\frac{dn}{d\tau} = \alpha_1 \frac{nc}{1+c} - n = f(n,c)$$

$$\frac{dc}{d\tau} = -\frac{nc}{1+c} - c + \alpha_2 = g(n,c)$$

where

n = bacterial concentration

c = chemical/nutrient concentration

We aim to prove that when a positive (physical) equilibrium exists the extinction state with $n=0$ is unstable.

- find the equilibria
- determine for what (α_1, α_2) parameter pairs the equilibria are positive/physical.
- Compute the Jacobian
- Use the TrA/detA classification to determine when each equilibria is stable.

Find the equilibria

(1) $$f(n, c) = n\left(\alpha_1 \frac{c}{c+1} - 1\right) = 0$$

(2) $$g(n, c) = -\frac{nc}{c+1} - c + \alpha_2 = 0$$

Extinction state

If $n = 0$ in (1)-(2) then $c = \alpha_2$

$$P_0 = (\bar{n}_0, \bar{c}_0) = (0, \alpha_2)$$

is always phsical if $\alpha_2 > 0$.

Coexistence state

Here we assume $n \neq 0$. Then (1) above $\Rightarrow$

(3) $$\alpha_1 \frac{c}{c+1} - 1 = 0$$

Solve this for c

(4) $$c = \frac{1}{\alpha_1 - 1}$$

Substitute (4) into ( ) and solve for n. The algebra is simpler than one might think since (3) implies

$$\frac{c}{c+1} = \frac{1}{\alpha_1}$$

Then $g(n, c) = 0$ becomes

$$(5) \qquad -n\left(\frac{1}{\alpha_1}\right) - \frac{1}{\alpha_1 - 1} + \alpha_2 = 0$$

Solve this to get

$$P_1 = (\bar{n}_1, \bar{c}_1) = \left(\alpha_1\left(\alpha_2 - \frac{1}{\alpha_1 - 1}\right), \frac{1}{\alpha_1 - 1}\right)$$

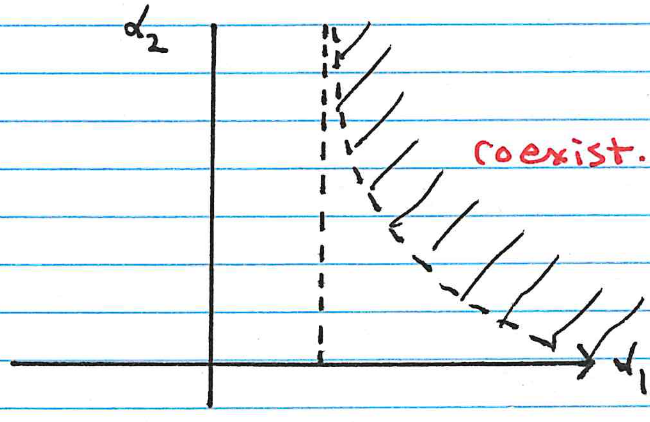
For $\bar{n}_1$ and $\bar{c}_1$ to be positive need $\alpha_1 > 1$ and $\alpha_2 > (\alpha_1 - 1)^{-1}$.

Summary

$P_0 = (0, \alpha_2)$ extinction state always exists

$P_1 = \left(\alpha_1\left(\alpha_2 - \frac{1}{\alpha_1 - 1}\right), \frac{1}{\alpha_1 - 1}\right)$ coexistence only if

$$\boxed{\begin{array}{l} \alpha_1 > 1 \\ \alpha_2 > \dfrac{1}{\alpha_1 - 1} \end{array}}$$



Jacobian (general)

$$\frac{dn}{d\tau} = f(n,c) = \alpha_1 \frac{nc}{1+c} - n$$

$$\frac{dc}{d\tau} = g(n,c) = -\frac{nc}{1+c} - c + \alpha_2$$

Recall Jacobian is

$$D\vec{F} = \begin{bmatrix} \frac{\partial f}{\partial n} & \frac{\partial f}{\partial c} \\ \frac{\partial g}{\partial n} & \frac{\partial g}{\partial c} \end{bmatrix}$$

Explicitly this is a bit of a mess

$$D\vec{F}(n,c) \equiv \begin{bmatrix} \alpha_1 c/((1+c) - 1 & \alpha_1 n/((1+c)^2 \\ -c/((1+c) & -n/((1+c)^2 - 1 \end{bmatrix}$$

Jacobian at P_1 coexistence

To simplify calculations define

$$A = \frac{\bar{n}_1}{(1+\bar{c}_1)^2}$$

After some calculations

$$D\vec{F}(P_1) = \begin{bmatrix} 0 & \alpha_1 A \\ -1/\alpha_1 & -(A+1) \end{bmatrix}$$

Despite the complexity of P_1 we get some simple results

$$\mathrm{Tr}\, D\vec{F} = -(A+1) \qquad <0 \text{ for stability}$$

$$\det D\vec{F} = A \qquad >0 \text{ for stability}$$

Then we have

$$\bar{n}_1 > 0 \Leftrightarrow A > 0 \Rightarrow P_1 \text{ stable}$$

In plain language

$$\boxed{P_1 \text{ stable} \Leftrightarrow P_1 \text{ physical}}$$

Jacobian at P_0 extinction

To simplify calculations

$$B = \frac{\alpha_2}{1+\alpha_2} > 0$$

After calculations

$$D\vec{F}(P_0) = \begin{bmatrix} \alpha_1 B - 1 & 0 \\ -B & -1 \end{bmatrix}$$

from which P_0 is stable only if

$$\mathrm{Tr}\, D\vec{F} = \alpha_1 B - 2 < 0$$

$$\det D\vec{F} = -(\alpha_1 B - 1) > 0$$

These two inequalities define a region in (α_1, α_2)-plane where P_0 stable. } part of HW

Can be used to show

P_0 stable only when P_1 not

or that no positive equilibria $\Rightarrow$ extinction.