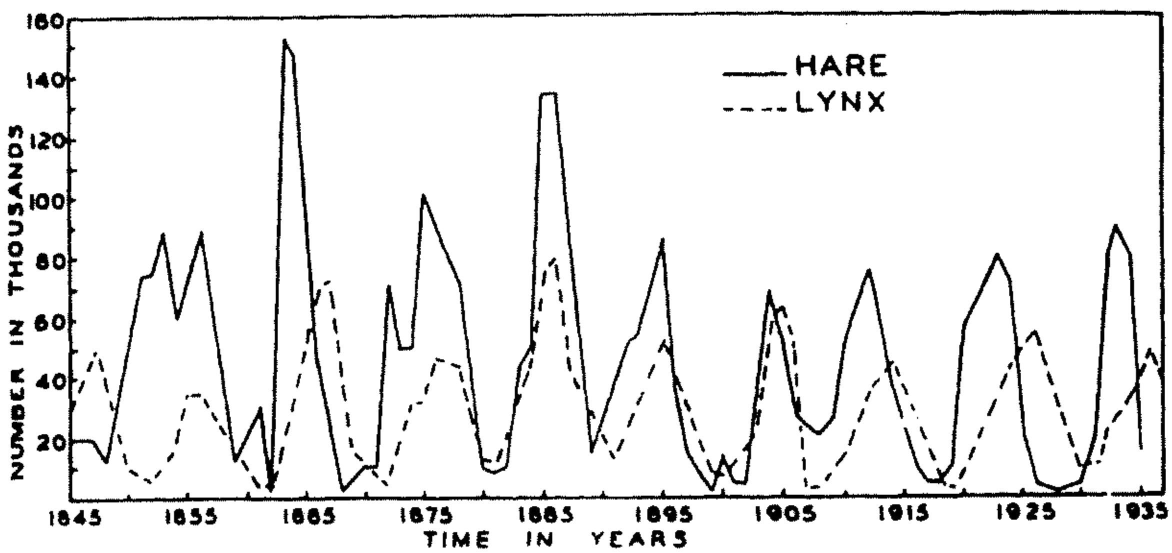
Predator Prey (Lotka Volterra) model



Model Equations

$$(1) \quad x' = ax - bxy$$

$$(2) \quad y' = -cy + dxy$$

Here

x = prey $\qquad [x] = \#$

y = predators $\qquad [y] = \#$

and predation terms proportional to xy

xy = number of possible interactions

Considerations: $b=0$, $d=0$ in model. Predators die out!

Model analysis

System has two equilibria

$$P_1(0,0) \qquad P_2\left(\frac{c}{d}, \frac{a}{b}\right)$$

Jacobian

$$D\vec{F}(P) = \begin{bmatrix} a - by & -bx \\ dy & dx - c \end{bmatrix}$$

Hence

$$D\vec{F}(P_1) = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} \qquad \det < 0 \qquad \text{saddle}$$

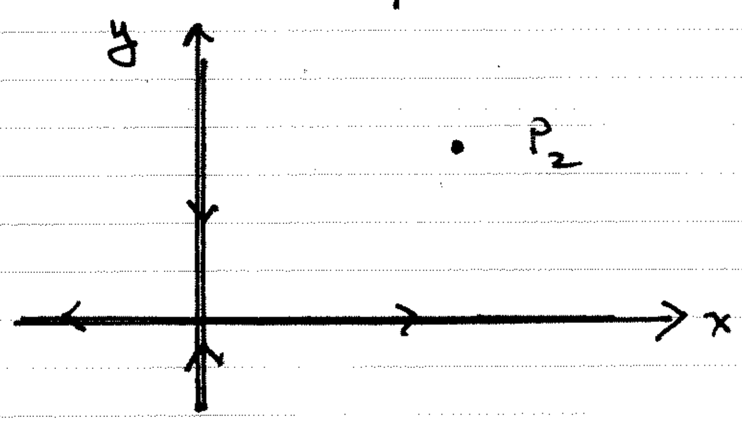
$$D\vec{F}(P_2) = \begin{bmatrix} 0 & -bc/d \\ ad/b & 0 \end{bmatrix} \qquad \begin{matrix} \det > 0 \\ Tr = 0 \end{matrix} \qquad \text{center}$$

For the saddle

$$\lambda_1 = a > 0 \qquad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{unstable}$$

$$\lambda_2 = -c < 0 \qquad \vec{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{stable}$$

In fact $x=0$ and $y=0$ are "invariant".



Since $\operatorname{Tr} DF(P_2) = 0$ it is not "hyperbolic" and linear theory may fail. Here linear theory predicts a center behavior near P_2. The validity of this remains to be shown:

First Integral

A function $\phi(x,y)$ is said to be a first integral if it is constant on trajectories.

A very lucky guess:

$$\phi(x,y) \equiv -c \ln x - a \ln y + dx + by \tag{3}$$

For the model (1)-(2) and some calculations

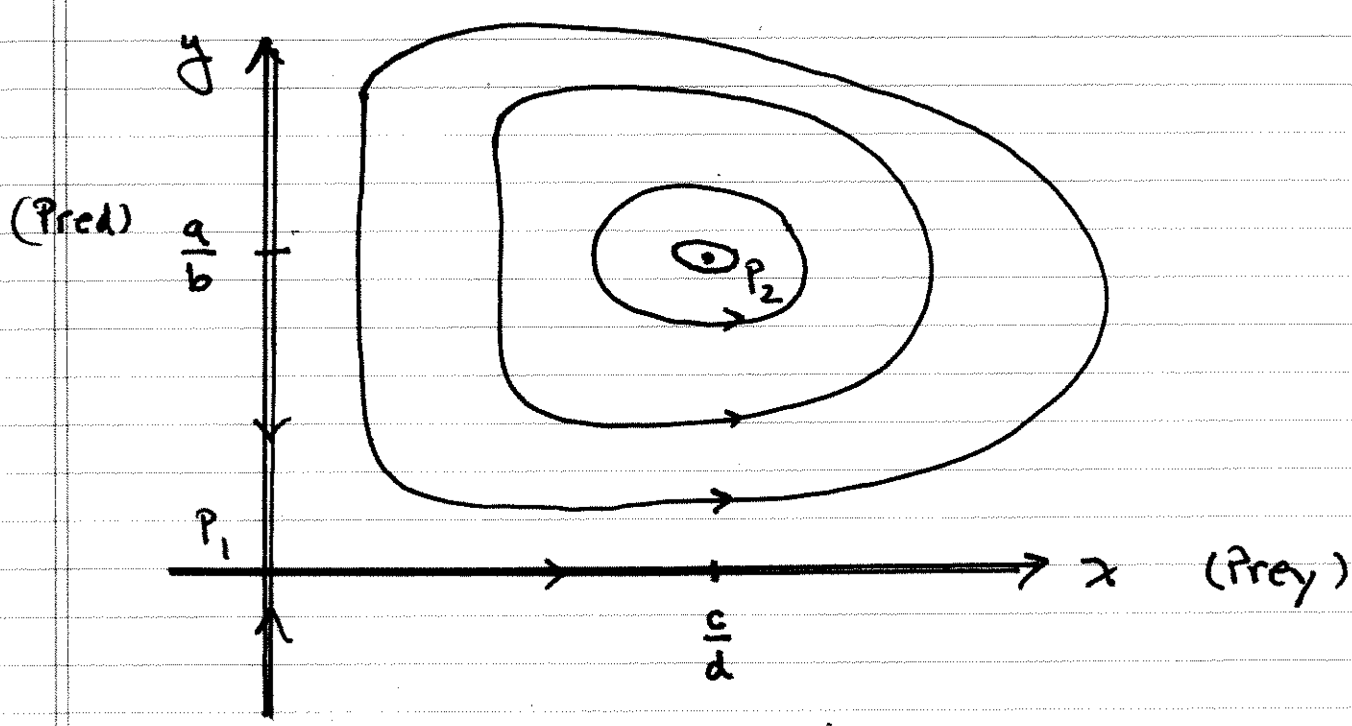
$$\frac{d\phi}{dt} = -\frac{c}{x} x' - \frac{a}{y} y' + dx' + by'$$

$$\frac{d\phi}{dt} = -\frac{c}{x}(ax - bxy) - \frac{a}{y}(-cy + dxy) + d(ax - bxy) + b(-cy + dxy)$$

$$\frac{d\phi}{dt} = 0$$

Thus, level sets of ϕ are trajectories of (1)-(2)

Need software to compute these level sets.



Neutrally stable concentric periodic ordits

Predator Prey with density dependence

Prey are often more numerous than predators. Hence, they are more subject to density growth rate effects. Predators are not. We examine such effects through a minor change in the previous model.

Model Equations (Logistic Growth Rate)

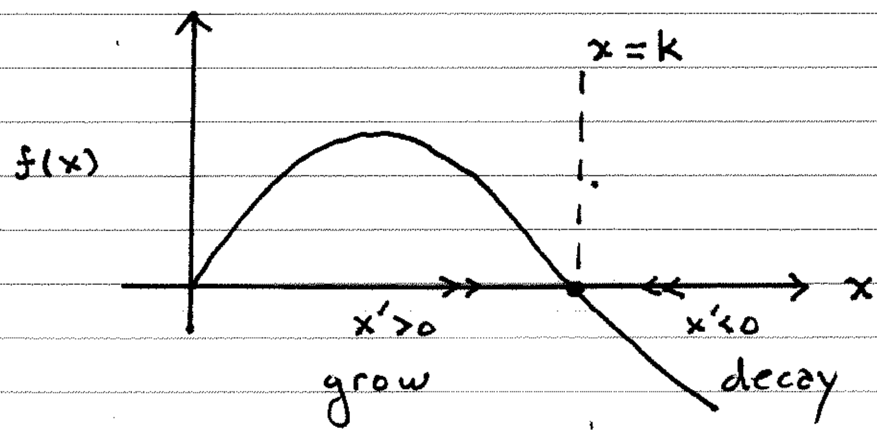
$$\boxed{\begin{aligned} x' &= ax\left(1-\frac{x}{K}\right) - bxy \\ y' &= -cy + dxy \end{aligned}}$$

(1) $x' = ax\left(1-\frac{x}{K}\right) - bxy$ prey

(2) $y' = -cy + dxy$ pred.

In the absense of predation ($y=0$)

$$x' = ax\left(1-\frac{x}{K}\right) = f(x)$$



Then $x(t) \to k$ called carrying capacity

Equilibria

$$0 = ax\left(1 - \frac{x}{K}\right) - bxy$$

$$0 = -cy + dxy$$

has three solutions (after calculations)

$P_1(0, 0)$ both extinct

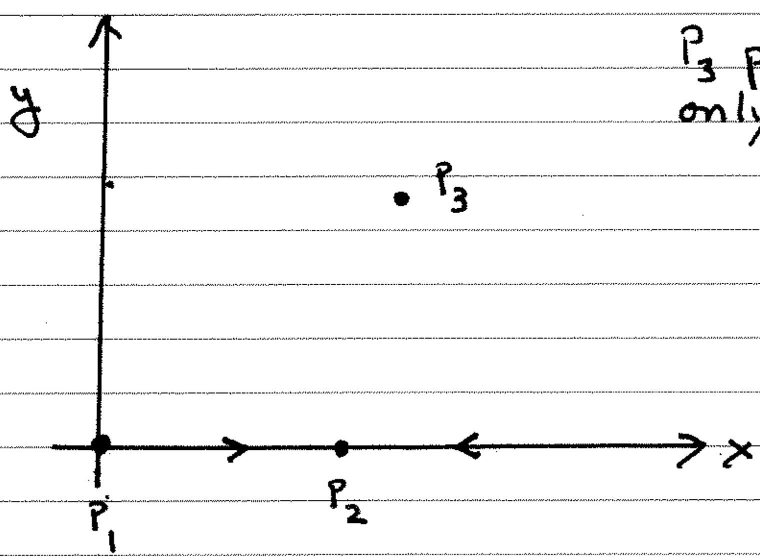
$P_2(k, 0)$ only pred. extinct

$P_3\left(\frac{c}{d}, \frac{a\Delta}{bkd}\right)$ coexistence

where

$$\boxed{\Delta = kd - c}$$

must be positive for a physical coexist P_3



P_3 physical only if $\Delta > 0$

Given P_3 the coexistence value of predators:

$$\text{pred. with density} = \left(\frac{\Delta}{Kd}\right) \times \text{pred. no density.}$$

where $\frac{\Delta}{Kd} = 1 - \frac{c}{Kd} > 0$ <u>smaller</u>!!

Jacobians at P_k (calculations omitted here)

$$DF_k = D\vec{F}(P_k)$$

Both extinct

$$DF_1 = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$$

$\det DF_1 = -ac < 0$

Saddle always

Pred. extinct

$$DF_2 = \begin{bmatrix} -a & -bk \\ 0 & \Delta \end{bmatrix}$$

$\det DF_2 = -a\Delta < 0$
when P_3 phsical $(\Delta > 0)$
Saddle.

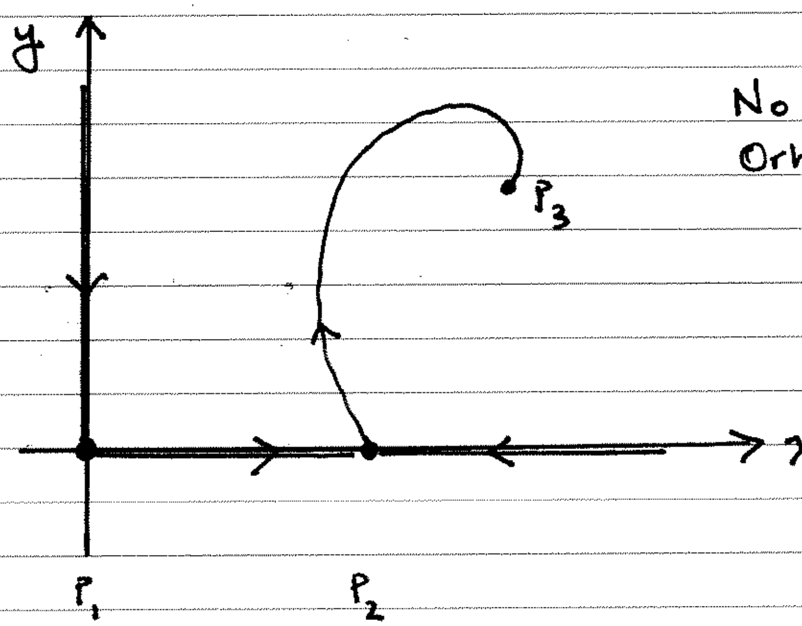
Coexistence

$$DF_3 = \begin{bmatrix} -\frac{ac}{kd} & -\frac{bc}{d} \\ \frac{a\Delta}{bk} & 0 \end{bmatrix}$$

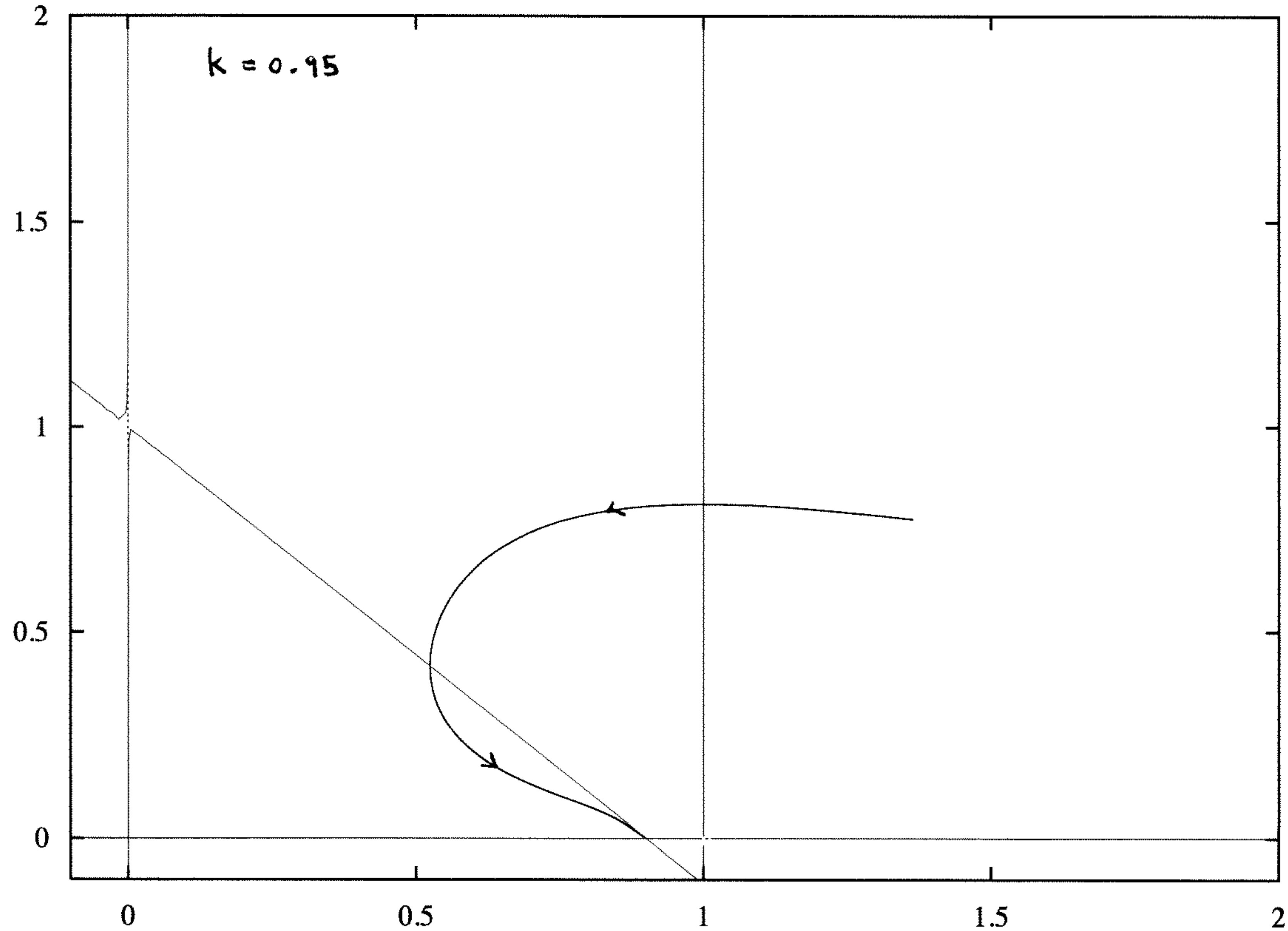
$\det DF_3 = \frac{ac\Delta}{dk}$

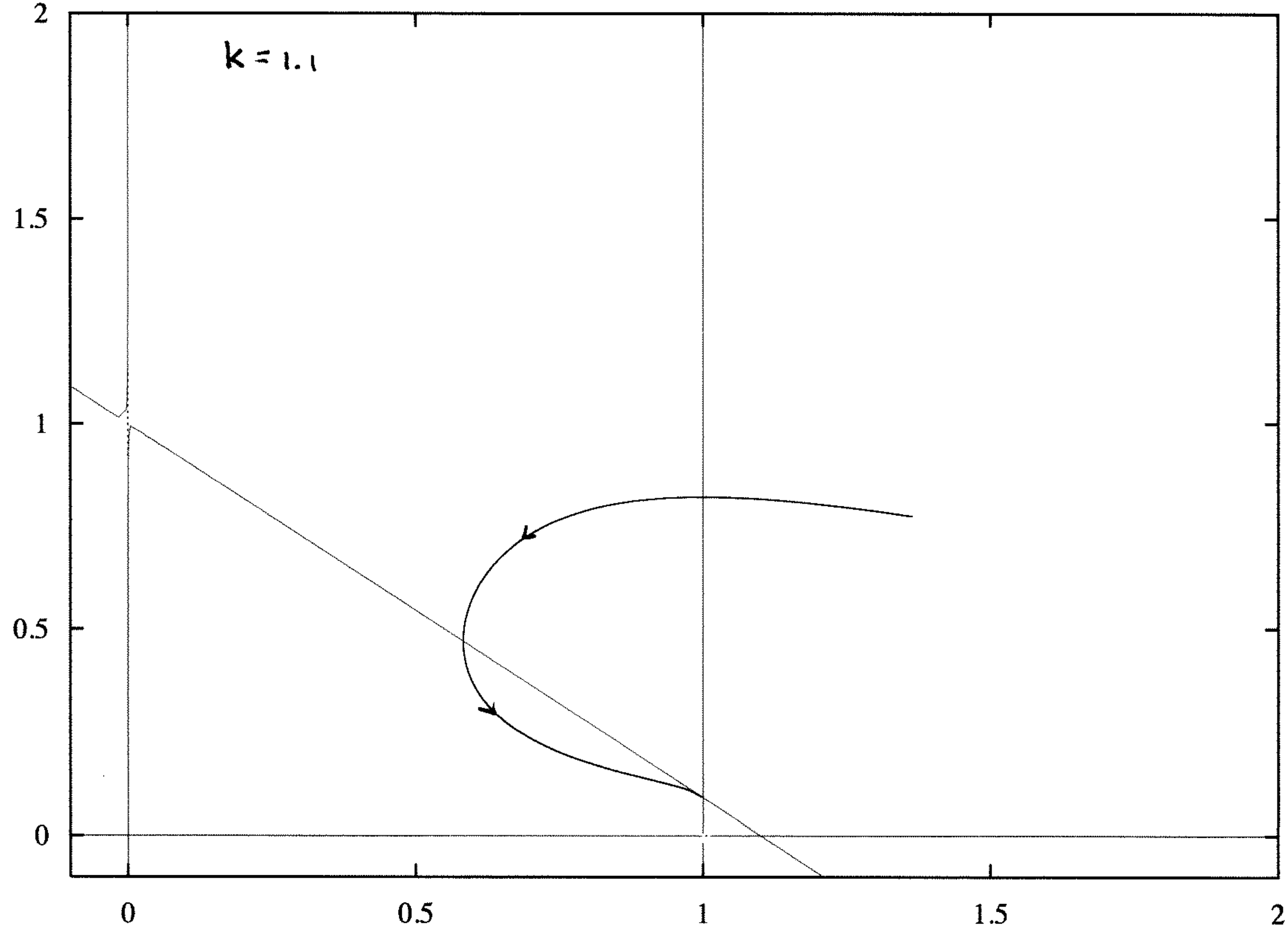
$\mathrm{Tr}\, DF_3 = \frac{-ac}{dk} < 0$

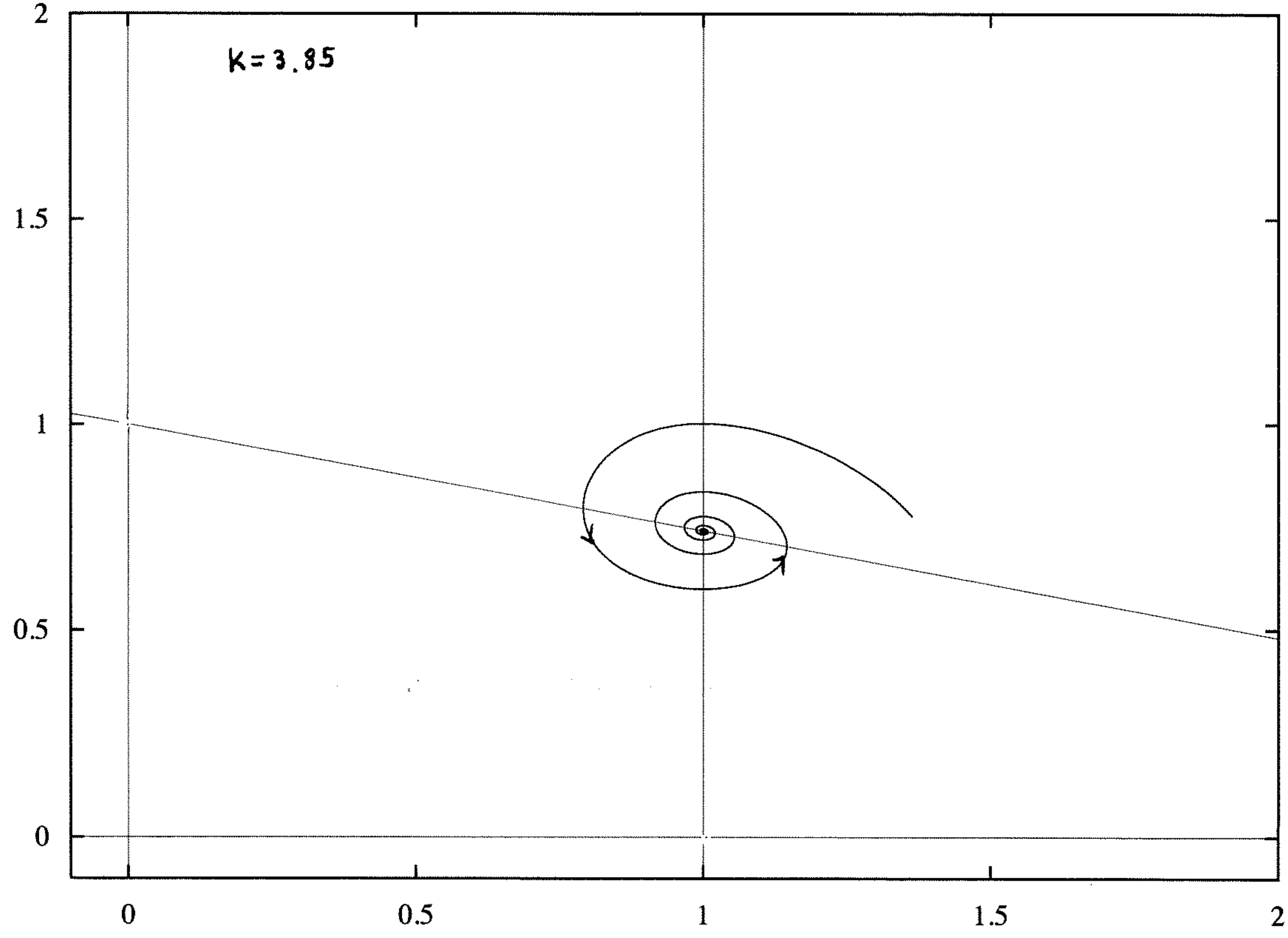
Hence P_3 is stable when physical.



No periodic Orbits.


k = 0.95
2
1.5
1
0.5
0
0
0.5
1
1.5
2


k = 1.1
2
1.5
1
0.5
0
0
0.5
1
1.5
2


K=3.85
2
1.5
1
0.5
0
0
0.5
1
1.5
2

```
> restart;
> with(linalg):
> f:=a*x*(1-x/k)-b*x*y:
> g:=-c*y+d*x*y:
> P:=solve({f=0,g=0},{x,y});
```

$$P := \{x=0, y=0\}, \{x=k, y=0\}, \left\{x=\frac{c}{d}, y=-\frac{a\,(-k\,d+c)}{b\,k\,d}\right\} \tag{1}$$

```
> DF:=map(simplify,jacobian([f,g],[x,y]));
```

$$DF := \begin{bmatrix} \frac{a\,k-2\,a\,x-b\,y\,k}{k} & -b\,x \\ y\,d & -c+d\,x \end{bmatrix} \tag{2}$$

```
> DF1:=subs(P[1],evalm(DF));det(DF1);
```

$$DF1 := \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$$

$$-a\,c \tag{3}$$

```
> DF2:=subs(P[2],evalm(DF));det(DF2);
```

$$DF2 := \begin{bmatrix} -a & -b\,k \\ 0 & k\,d-c \end{bmatrix}$$

$$-a\,(k\,d-c) \tag{4}$$

```
> DF3:=map(simplify,subs(P[3],evalm(DF)));det(DF3);trace(DF3);
```

$$DF3 := \begin{bmatrix} -\frac{a\,c}{k\,d} & -\frac{b\,c}{d} \\ -\frac{a\,(-k\,d+c)}{b\,k} & 0 \end{bmatrix}$$

$$-\frac{c\,a\,(-k\,d+c)}{d\,k}$$

$$-\frac{a\,c}{k\,d} \tag{5}$$