Competition Models

Two species (or more) competing for the same resources

Though they share the same habitat, their carrying capacities K_i differ.

$$\boxed{\begin{aligned} \frac{dN_1}{dt} &= r_1 N_1 (K_1 - a_{11} N_1 - a_{12} N_2) \\ \frac{dN_2}{dt} &= r_2 N_2 (K_2 - \underbrace{a_{21} N_1 - a_{22} N_2}) \end{aligned}}$$

Note the density dependence weighted average growth rate (underbrace above)

Rather than analyze model directly (8 parameters) we nondimensionalize.

$$x = \frac{N_1}{N_1^*} \qquad y = \frac{N_2}{N_2^*} \qquad \tau = \frac{t}{t^*}$$

Excluding lengthy calculations we use

$$N_1^* = \frac{K_1}{a_{11}} \qquad N_2^* = \frac{K_2}{a_{22}} \qquad t^* = \frac{1}{r_1 K_1}$$

Dimensionless Model

$$\boxed{\begin{aligned} \dot{x} &= f(x,y) = x(1-x-by) \\ \dot{y} &= g(x,y) = \eta y(1-cx-y) \end{aligned}}$$

where

$$b = \frac{a_{12}K_2}{a_{22}K_1} \qquad c = \frac{a_{21}K_1}{a_{11}K_2} \qquad \eta = \frac{r_2 K_2}{r_1 K_1}$$

The dimensionless model has only 3 params!

Equilibria

Solving $f = g = 0$ we find four possible equilibria

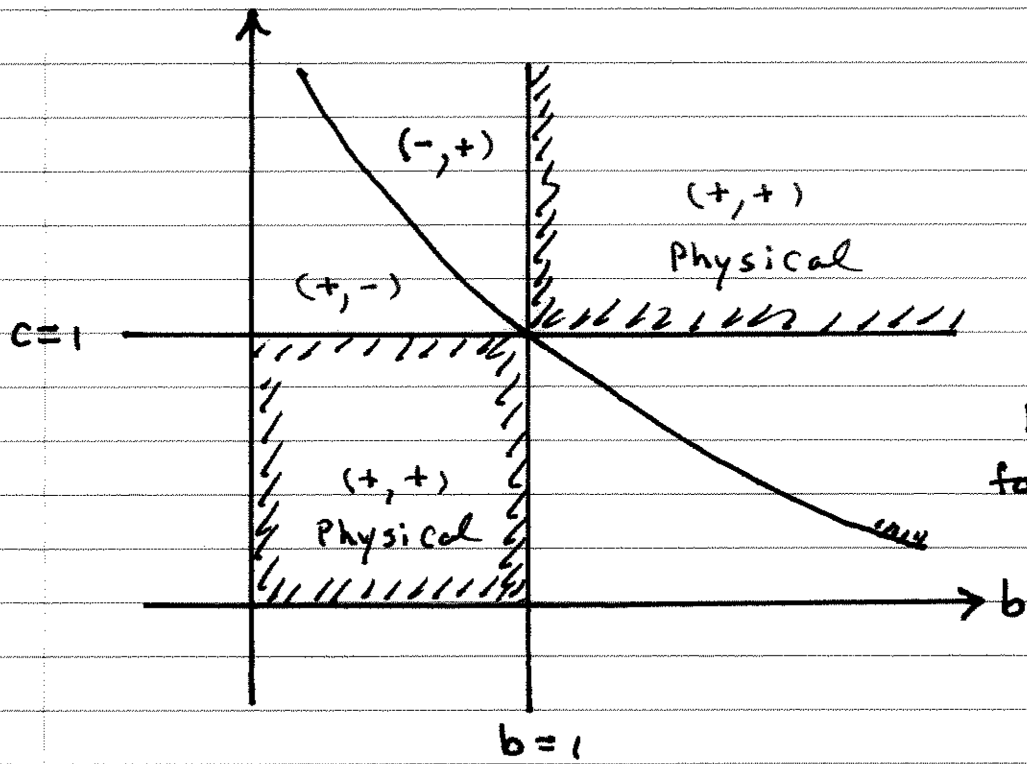
$P_0 = (0, 0)$	both die
$P_x = (1, 0)$	x survives
$P_y = (0, 1)$	y survives
$P_1 = \left(\frac{b-1}{bc-1}, \frac{c-1}{bc-1}\right)$	possible coexistence.

Existence of positive interior P_1

The coexistence state P_1 is positive (physical) if both coefficients of P_1 are positive:

$$P_1 = \left(\frac{b-1}{bc-1}, \frac{c-1}{bc-1} \right)$$

Plot $b=1$, $c=1$ and $c = 1/b$



Shows sign of components of P_1 in (b, c)-plane.

$$\boxed{P_1 \text{ physical only in shaded regions}}$$

Stability Conditions

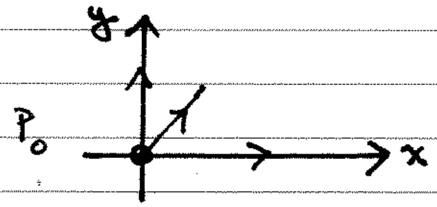
Must evaluate Jacobian at each equilibria.

$$DF(x,y) = \begin{bmatrix} 1-2x-by & -xb \\ -\eta y c & \eta - \eta c x - 2\eta y \end{bmatrix}$$

Total Extinction P_0

$$DF(P_0) = \begin{bmatrix} 1 & 0 \\ 0 & \eta \end{bmatrix}$$

$\lambda = 1, \eta$ unstable node with axes as trajectories

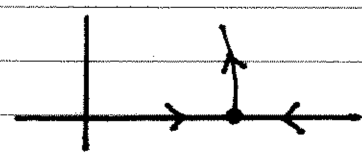


Species x survives P_x

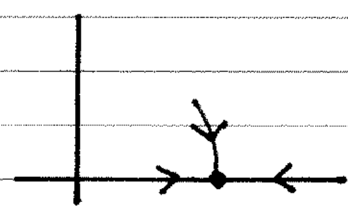
$$DF(P_x) = \begin{bmatrix} -1 & -b \\ 0 & \eta(1-c) \end{bmatrix}$$

$\lambda = -1, \eta(1-c)$

$c < 1$ P_x saddle



$c > 1$ P_x stable

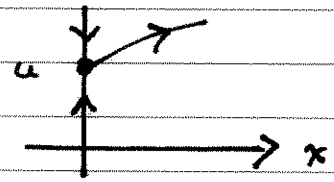


Species y survives $\underrightarrow{\quad}$ P_y

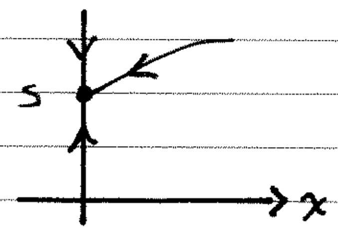
$$DF(P_y) = \begin{bmatrix} 1-b & 0 \\ -\eta c & -\eta \end{bmatrix} \qquad \lambda = 1-b, -\eta$$

From the eigenvalues we conclude

$b < 1$ $\quad$ P_y saddle



$b > 1$ $\quad$ P_y stable

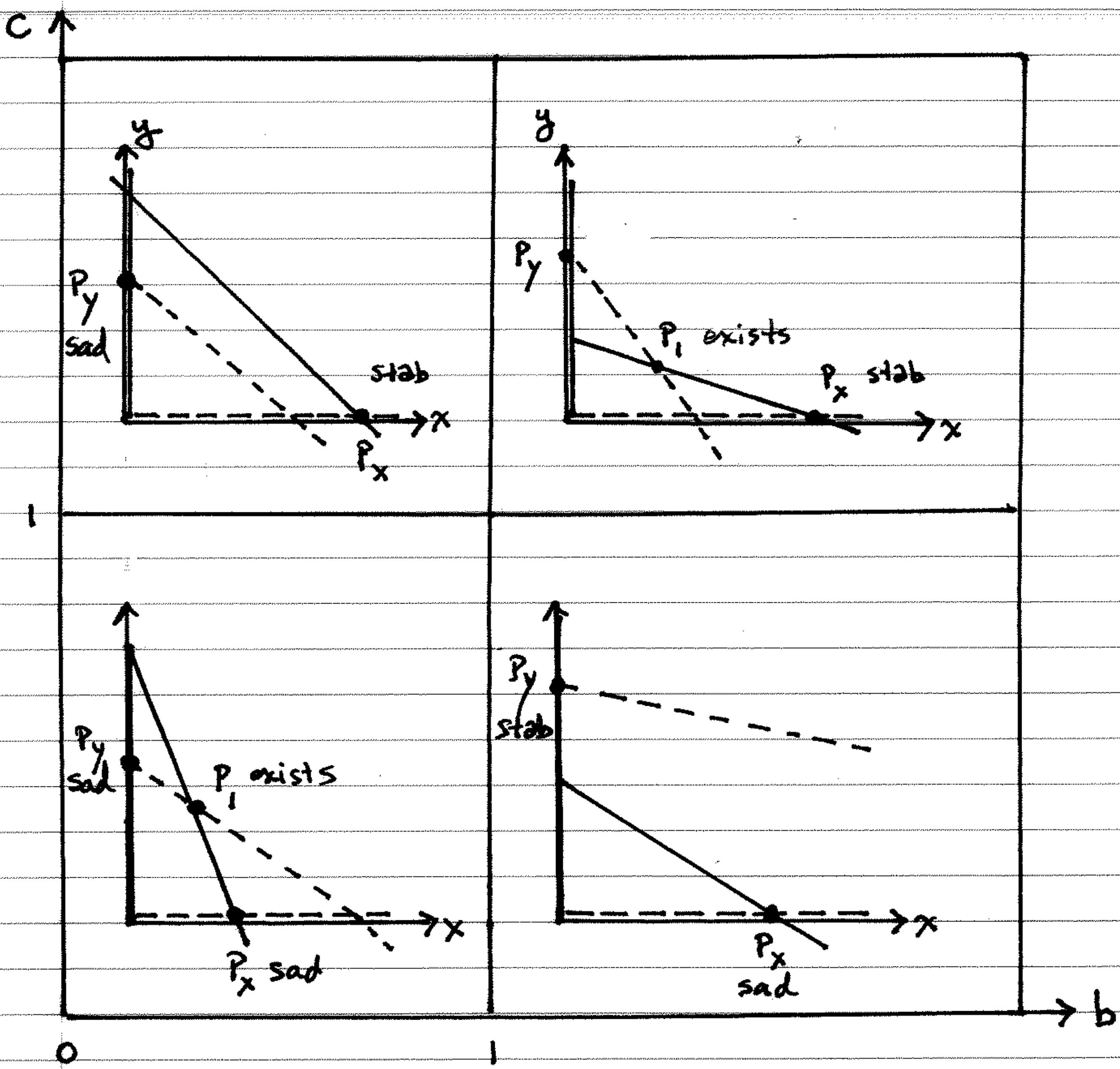


Nullclines and Equilibria Existence

$$\dot{x} = 0 \quad \Leftrightarrow \quad x = 0, \; y = \tfrac{1}{b}(1-x) \qquad \text{———}$$

$$\dot{y} = 0 \quad \Leftrightarrow \quad y = 0, \; y = 1 - cx \qquad \text{- - - - -}$$

Below are nullclines of system for (b, c)-pairs



To completely understand what happens for all (b, c) need to examine stability of P_1 when it exists.

Coexistence Equilibria P_1

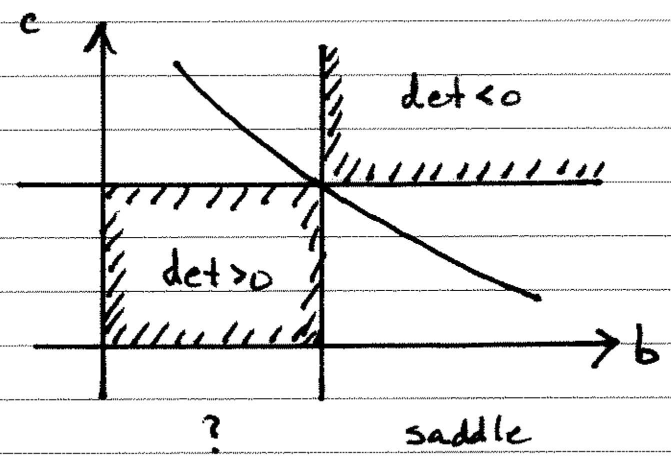
One can calculate the Jacobian at P_1 :

$$DF(P_1) = \frac{1}{(1-bc)} \begin{bmatrix} b-1 & b(b-1) \\ \eta c(c-1) & \eta(c-1) \end{bmatrix}$$

After some calculations

$$\det DF(P_1) = \frac{\eta(b-1)(c-1)}{1-bc}$$

the sign of which can be deduced in the (b,c)-plane



P_1 exists only in shaded regions

So, for $b>1$, $c>1$ we know P_1 exists and is a saddle.

To ascertain the stability when $b<1$, $c<1$ we need to examine the Trace of $DF(P_1)$.

Stability of coexistence $(b,c) \in (0,1)^2$

For
$$0 < b < 1 \qquad 0 < c < 1$$

we found $\det DF > 0$ so that equilibria P_1 is not a saddle. Therefore, its stability depends entirely on

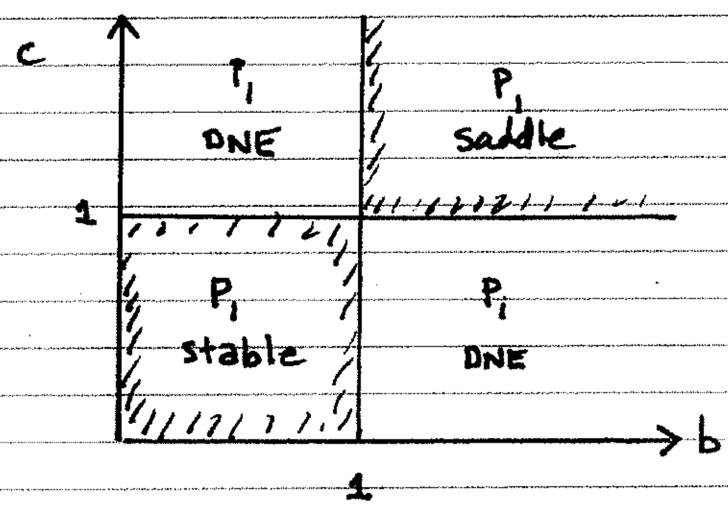
$$\mathrm{Tr}\, DF(P_1) = \frac{(b-1) + \eta(c-1)}{1-bc}$$

For the parameter range being discussed the numerator is negative while the denominator is positive. Hence

$$\mathrm{Tr}\, DF(P_1) < 0 \qquad \forall \eta$$

P_1 stable

Summary of P_1 existence and stability:



Conclusion of Phase Portraits