Infectious Disease Models (SIR)

S = susceptible

I = infected

R = removed

Simplest model

$$\boxed{S} \xrightarrow{\alpha} \boxed{I} \xrightarrow{\beta} \boxed{R}$$

Assume

(A1) Susceptibles only get infected by coming in contact with infected with probability of contraction proportional to SI

(A2) Infected recover at rate βI

$$\boxed{\begin{aligned} \frac{dS}{dt} &= -\alpha S I \\ \frac{dI}{dt} &= \alpha S I - \beta I \\ \frac{dR}{dt} &= \beta I \end{aligned}}$$

By adding these equations we can see the total population N is constant:

$$N = S + I + R$$

Although the SIR model has three dependent variables, the first two are decoupled so the dynamics is determined by

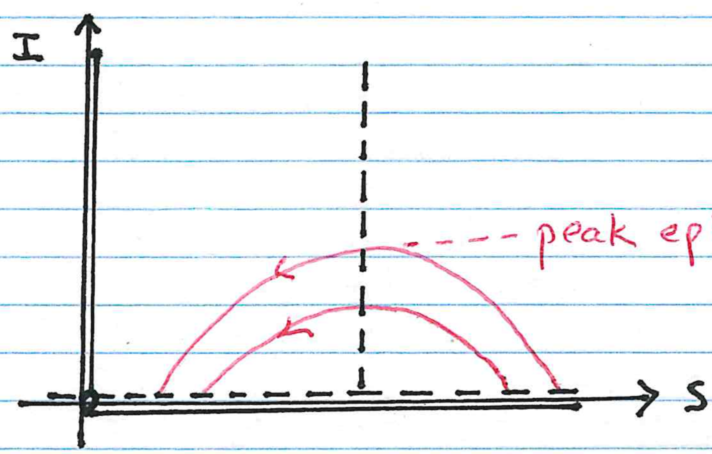
$$\boxed{\begin{aligned}\frac{dS}{dt} &= -\alpha S I \\ \frac{dI}{dt} &= \alpha S I - \beta I\end{aligned}}$$

—— x nullcline

- - - - y nullcline

Phase Portrait

System has a line of fixed points



line of equilibria on $I=0$ where $\dot{x}=0$ and $\dot{y}=0$

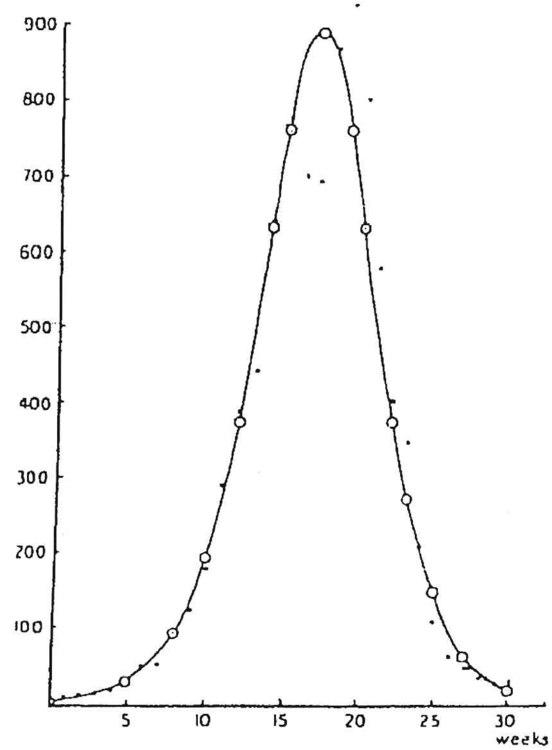
Since

$$I + S + R = N \qquad \text{(total population)}$$

is constant, the SIR model only models epidemics when no one dies.

Also for the rate at which cases are removed by death or recovery which is the form in which many statistics are given

$$\frac{dz}{dt} = \frac{l^3}{2x_0\kappa^2}\sqrt{-q}\,\mathrm{sech}^2\left(\frac{\sqrt{-q}}{2}lt - \phi\right). \qquad \text{(thirty-one)}$$



The accompanying chart is based upon figures of deaths from plague in the island of Bombay over the period December 17, 1905, to July 21, 1906. The ordinate represents the number of deaths per week, and the abscissa denotes the time in weeks. As at least 80 to 90 per cent. of the cases reported terminate fatally, the ordinate may be taken as approximately representing dz/dt as a function of t. The calculated curve is drawn from the formula

$$\frac{dz}{dt} = 890\,\mathrm{sech}^2(0\cdot 2t - 3\cdot 4).$$

$$\frac{dz}{dt} = \beta I$$

Figure 6.13 On a page from their original article, Kermack and McKendrick compare predictions of the model given by equations (27a,b,c) with data for the rate of removal by death. Note: dz/dt *is equivalent to* dR/dt *in equations (27).[Kermack, W. O., and McKendrick, A. G. (1927). A contribution to mathematical theory of epidemics.* Roy Stat. Soc. J., 115, *714.]*

Analytic solution of SIR model

Without loss of generality $\alpha = 1$ else re-scale time. From ODEs

$$\frac{d}{dt}\left(\ln S\right) = -I = -\frac{1}{\beta}\frac{dR}{dt}$$

$$\frac{d}{dt}(\ln S) = -\frac{1}{\beta}\frac{d}{dt}(N - S - I)$$

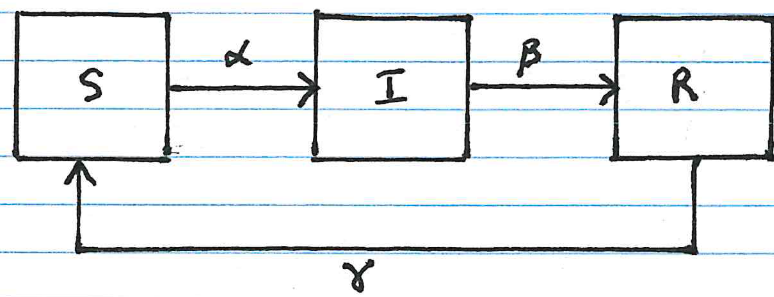
$$\frac{d}{dt}(\ln S) = \frac{d}{dt}\left(\frac{S + I}{\beta}\right)$$

integrate and some algebra:

$$\boxed{\ln S^{\beta} = I + S + c}$$

where constant c detemined by initial conditions.

SIRS model (Temporary immunity)



In the SIRS model recovered individuals can become re-infected.

$$\frac{dS}{dt} = -\alpha S I + \gamma R$$

$$\frac{dI}{dt} = \alpha S I - \beta I$$

$$\frac{dR}{dt} = \beta I - \gamma R$$

By adding these three equations we see the ~~fot~~ total population remains constant

$$N = S + I + R \qquad \text{(no deaths)}$$

Use this to eliminate R:

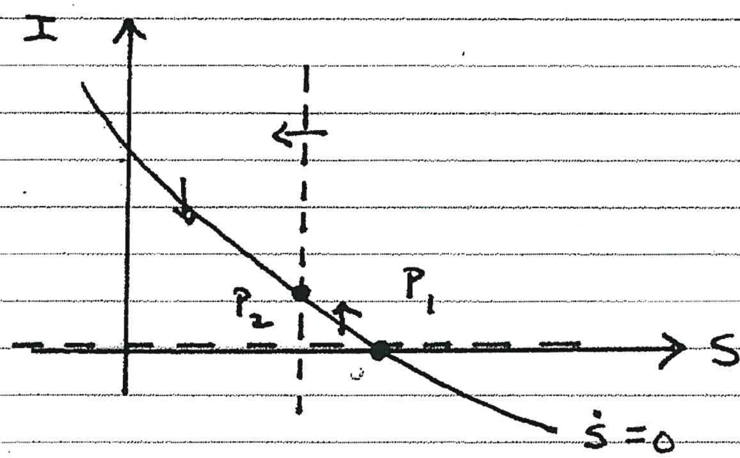
$$\boxed{\begin{aligned} \frac{dS}{dt} &= f(S, I) = -\alpha S I + \gamma (N - S - I) \\ \frac{dI}{dt} &= g(S, I) = \alpha S I - \beta I \end{aligned}}$$

Nullclines and Equilibria

$\dot{I} = 0 \qquad I = 0 \qquad S = \beta/\alpha$

$\dot{S} = 0 \qquad I = \dfrac{\gamma(N-S)}{\alpha S + \gamma}$

Rough phase portrait for certain (α, β, γ)



has two equilibria $P_k = (S_k, I_k)$ where

$$P_1 = (N, 0)$$

$$P_2 = \left(\frac{\beta}{\alpha}, \frac{\gamma(N - S_2)}{\alpha S_2 + \gamma} \right)$$

Written another way the I component of P_2 is

$$I_2 = \frac{\gamma(N\alpha - \beta)}{\alpha(\beta + \gamma)} > 0$$

$$\boxed{\text{Equilibria } P_2 \text{ is physical only if } N > \beta/\alpha}$$

Equilibria stability

$$DF = \begin{bmatrix} -\alpha I - \gamma & -\alpha S - \gamma \\ \alpha I & \alpha S - \beta \end{bmatrix}$$

Equilibria $P_1 = (N, 0)$

$$\det DF(P_1) = -\gamma(N\alpha - \beta)$$

$$\operatorname{Tr} DF(P_1) = -\gamma + (N\alpha - \beta)$$

From these we deduce

P_2 physical $\Rightarrow$ P_1 saddle.

P_2 not physical $\Rightarrow$ P_1 stable

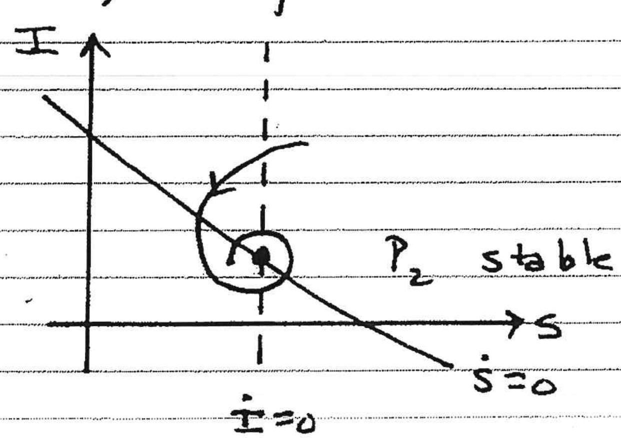
Equilibria $P_2 = (S_2, I_2)$

After some calculations (assuming P_2 physical)

$$\det DF(P_2) = \alpha I_2(\beta + \gamma) > 0$$

$$\operatorname{Tr} DF(P_2) = -(\alpha I_2 + \gamma) < 0$$

hence stable, with potential oscillations



$N > \beta/\alpha$

Table 6.1 **A Summary of Several Epidemic Models**

Type	*Immunity*	*Birth/Death*	*Significant quantity*	*Results*
SIS	None	Rate $= \delta$	$\sigma = \dfrac{\beta}{\gamma + \delta}$	(1) $\sigma > 1$: constant endemic infection (2) $\sigma < 1$: infection disappears
		Additional disease fatality rate η	σ as above, and $\epsilon = \dfrac{\eta}{\gamma + \delta}$	Disease always eventually disappears leaving some susceptibles.
SIR	Yes, recovery gives immunity.	None	$\sigma = \dfrac{S_0 \beta}{\gamma}$ (S_0 = initial S)	(1) $\sigma > 1$: infection peaks and then disappears (2) $\sigma < 1$: infection disappears
		Yes, rate $= \delta$	$\sigma = \dfrac{\beta}{\nu + \delta}$	(1) $\sigma < 1$: susceptibles and infectives approach constant levels (2) $\sigma < 1$: infectives disappear; only S remains
(*SIR* with carriers)	Yes	Yes		Disease always remains endemic.
SIRS	Temporary, lost at rate γ	Rate $= \delta$	$\sigma = \dfrac{\beta}{\nu + \delta}$	(1) $\sigma > 1$: same as *SIR* (1) but higher levels of infectives (2) $\sigma < 1$: same as *SIR* (2)

Table 6.2 ***Estimates of the intrinsic reproductive rate R_0 for human diseases and the corresponding percentage of the population p that must be protected by immunization to achieve eradication. [Reprinted by permission, American Scientist,*** **journal of Sigma Xi,** ***"Parasitic Infections as Regulator of Animal Populations," by Robert M. May, 71:36–45 (1983).]***

Infection	*Location and Time*	R_0	*Approximate Value of p (%)*
Smallpox	Developing countries, before global campaign	3–5	70–80
Measles	England and Wales, 1956–68;	13	92
	U.S., various places, 1910–30	12–13	92
Whooping cough	England and Wales, 1942–50;	17	94
	Maryland, U.S., 1908–17	13	92
German measles	England and Wales, 1979;	6	83
	West Germany, 1972	7	86
Chicken pox	U.S., various places, 1913–21 and 1943	9–10	90
Diphtheria	U.S., various places, 1910–47	4–6	~80
Scarlet fever	U S., various places, 1910–20	5–7	~80
Mumps	U.S., various places, 1912–16 and 1943	4–7	~80
Poliomyelitis	Holland, 1960: U.S., 1955	6	83

The fraction p to be immunized is then deduced from the following simple calculation:

Intrinsic reproductive rate of disease $= R_0$, Fraction immunized $= p$, Fraction not immunized $= 1 - p$, Population participating in disease $= N(1 - p)$, Effective intrinsic reproduction rate of disease (after immunization) $= R_0' = (1 - p)R_0$.

Thus

$$R_0' < 1 \Rightarrow (1 - p)R_0 < 1 \Rightarrow p > 1 - \frac{1}{R_0}$$

Other Compartmental models

M = Passively immune infants
S = susceptibles
E = exposed in latent period
I = infected
R = recoverd with immunity
V = vaccinated individuals

For example one SEIR model $N = S + E + I + R$

$$\dot{S} = \mu(N - S) - (\beta \cdot \tfrac{1}{N}) I S$$

$$\dot{E} = (\beta \cdot \tfrac{1}{N}) S - (\mu + a) E$$

$$\dot{I} = a E - (\nu + \mu) I$$

$$\dot{R} = \nu I - \mu R$$

Here all E, I, R states can become susceptible

Can verify $\dot{N} = 0$.