Math 450 (2017) – Final (Take home)

Due: Friday, December 8, 2017 (10am) NAME: ______________________

Get the exam no me no later than 10am on December 8, 2017. You may give me the exam in person at class or slide under my office door (Wil 2-136). You may not talk to other students but may use the text , my notes or you can ask me clarifying questions.

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1. [25 pts] Let $y(t, \epsilon)$ be the solution of the initial value problem

$$\begin{aligned} y'' + y &= \epsilon y^5 \quad , \quad 0 < \epsilon \ll 1 \\ y(0) &= 0 \quad , \quad y'(0) = 1 \end{aligned}$$

where ( )$'$ denotes differentiation in t. Assume

$$\begin{aligned} y(t, \epsilon) &= y_0(\tau) + \epsilon y_1(\tau) + O(\epsilon^2) \\ \tau &= \omega(\epsilon)t \equiv (1 + \omega_1\epsilon + \omega_2\epsilon^2 + \cdots)t \end{aligned}$$

Use Poincare-Lindstedt's method to determine ω_1 and the $O(\epsilon)$ correction to the period of the oscillation. You may use the identity:

$$\sin^5 A = \frac{5}{8}\sin A - \frac{5}{16}\sin 3A + \frac{1}{16}\sin 5A$$

2. [25 pts] The following equation has two roots for positive ϵ.

$$\epsilon x^4 + \frac{1}{\sqrt{x}} = x \quad , \quad 0 < \epsilon \ll 1$$

Find a two term expansion in ϵ for the singular root $x = \bar{x}(\epsilon)$. Make sure you balance the largest two terms. Also, you may use the binomial expansion:

$$(X_0 + \delta X_1 + \cdots)^p = X_0^p + p\ X_0^{p-1}X_1\ \delta + O(\delta^2) \quad , \quad \delta \ll 1$$

3. [25pts] Let $y(x,\epsilon)$ be the solution of the following boundary value problem:

$$\begin{aligned} \epsilon y'' + (x+2)y' + y^2 &= 0 \quad , \quad x \in (0,1) \quad , \quad 0 < \epsilon \ll 1 \\ y(0) &= A \quad , \quad y(1) = \frac{1}{ln(3)} \end{aligned}$$

a) Find a uniformly valid approximation $y_u(x,\epsilon)$ of the solution for arbitrary A.

b) For what value of A is there no boundary layer at $x = 0$? This happens when the outer solution satisfies both boundary conditions.

4. [25 pts] A functional $J : \mathcal{A} \to \mathbb{R}$ is defined by

$$\begin{aligned} J(y) &= \int_0^1 L(x, y(x), y'(x))\ dx \\ \mathcal{A} &= \{y \in C^2[0,1] : y(0) = 2, y(1) = 3\} \end{aligned}$$

where the Lagrangian

$$L(x, y, y') = y\ y'\ ln(y')$$

Use a first integral of the Euler-Lagrange equations to find the extremal $\bar{y} \in \mathcal{A}$ of the functional J. You may assume $\bar{y}$ and $\bar{y}'$ are not negative.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \; dx \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; dx$$