Math 450 (2023) – Final (Take home)

Due: Monday, December 11, 2023 (10-11:50am) NAME: ______________________

deliver in person in Wil 1-126

Instructions You may use your notes, posted class notes, the textbook and any differential equations textbook. You may not use software. Staple this sheet to your work when you turn it in (in class) on Friday 10/20/23. You must work alone but may ask me a clarifying question. Give me your exam in person in class (Wil 1-126) on Mond. Dec 11, 10-11:50am. Lastly note this exam is out of 50 points.

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1. [10 pts] The following equation has three real roots for small positive ϵ.

$$\epsilon x^2 + \frac{2}{1+x} = 1 \quad , \quad 0 < \epsilon \ll 1 \tag{1}$$

Find two term expansions for **both** of the singular roots:

$$x = \frac{X}{\epsilon^\alpha} = \frac{1}{\epsilon^\alpha}\left(X_0 + \delta X_1 + o(\delta)\right) \qquad , \qquad 0 < \delta \ll 1$$

Specifically what are α, X_0, δ and X_1. Also, when finding α, make sure you balance the two **largest** of (1) terms noting that

$$\frac{1}{1+x} = \frac{\epsilon^\alpha}{\epsilon^\alpha + X} \ll 1$$

2. [15pts] Let $y(x,\epsilon)$ be the solution of the following boundary value problem:

$$\begin{aligned} \epsilon\sqrt{x}\, y'' + \sqrt{x}\, y' + y^2 &= 0 \quad , \quad x \in (0,1) \quad , \quad 0 < \epsilon \ll 1 \\ y(0) &= 1 \quad , \quad y(1) = 2 \end{aligned}$$

a) Find the leading order outer approximation $y_0(x)$ satisfying the right boundary condition $y_0(1) = 2$. The equation is separable.

b) Find the boundary layer thickness $\delta(\epsilon)$ at $x = 0$ and the leading order inner approximation $Y_0(X)$ in the inner expansion

$$y(x,\epsilon) = Y(X,\epsilon) = Y_0(X) + o(1) \qquad , \qquad X \equiv \frac{x}{\delta}$$

that satifies $Y_0(0) = 1$

c) Use matching to find a uniformly valid approximation $y_u(x,\epsilon)$ of the solution for the problem.

3. [15 pts] A functional $J : \mathcal{A} \to \mathbb{R}$ is defined by

$$\begin{aligned} J(y) &= \int_0^\pi L(y, y')\, dx \\ \mathcal{A} &= \{y \in C^2[0,\pi] : y(0) = 1, y(\pi/4) = \pi/16\} \end{aligned}$$

where the Lagrangian

$$L(y, y') = 4y^2 - (y')^2 + x^2 y'$$

Find the extrema $\bar{y} \in \mathcal{A}$ of the functional J. DO NOT use a first integral to find the extrema. The Euler-Lagrange equation as is should be a simple linear equation for $\bar{y}$.

4. [10 pts] As discussed in class and the notes, the displacement $y(x)$ of a loaded elastic beam minimizes the potential energy of the beam:

$$J(y) = \int_0^1 \left(\frac{1}{2}\mu (y'')^2 - py \right)\, dx$$

Here we shall assume the flexural rigidity $\mu = 1$ and the load $p = 12$ is constant. The beam is fixed to pins in the wall so that

$$y(0) = 0 \quad , \quad y(1) = 0 \ .$$

Find the two Natural Boundary Conditions for the problem, the Euler Lagrange equations and then find the extrema that minimizes $J(y)$.