1. [30pts] A fluid of density $\rho$ exerts a (drag) force $F$ on a cylinder of diameter $D$ as it flows around it. The fluid has viscosity $\mu$ and the fluid velocity far from the cylinder is $v$. Assume the physical law

$$f(\rho, F, D, \mu, v) = 0$$

and then find all the dimensionless $\pi$ of the form

$$\pi = \rho^{\alpha_1} F^{\alpha_2} D^{\alpha_3} \mu^{\alpha_4} v^{\alpha_5}$$

Note that the units of viscosity are $[\mu] = ML^{-1}T^{-1}$.

2. [20pts] In quantum mechanics the wave function $\psi(X)$ of a particle of mass $m$ is a solution of Schrödinger's equation. For the quantum harmonic oscillator problem, the (time-independent) Schrödinger equation is:

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dX^2} + \frac{1}{2} m\omega^2 X^2 \psi = E\psi$$

where $X$ is position, $\omega$ is the frequency of the oscillator (potential well), $\hbar$ is Planck's constant and $E$ is the energy of the system. The wave function $\psi$ depends on position $X$ but since it is a probability density function (used to determine the location of the particle) it is dimensionless and hence $[\psi] = 1$. Note here that $[\omega] = T^{-1}$ where $T$ is time.

i) Determine the units of $\hbar$ in terms of $M, T, L$

ii) Nondimensionalize the problem so it has only one dimensionless quantity, i.e.

$$-\frac{d^2 \psi}{dx^2} + (x^2 - E)\psi = 0$$

Note: Since $\psi$ is already dimensionless, you only need to rescale the variable $X$:

$$x = \frac{X}{X^*}$$

Determine the parameter $X^*$ which will yield the dimensionless equation in (1). Clearly express $X^*$ and the dimensionless parameter $E$ in terms of the original dimensional parameters $m, \omega, \hbar$ and $E$. 
3. [25pts] The equation

\[ f(x, \epsilon) = x - \frac{1}{(x + \epsilon)^3} = 0, \quad 0 < \epsilon \ll 1 \]

has two real roots \( \bar{x}_\pm(\epsilon) \). Assume

\[ \bar{x}(\epsilon) = x_0 + x_1 \epsilon + O(\epsilon^2) \]

and then determine \( x_0 \) and \( x_1 \) for both roots. If you use the Binomial theorem note

\[ (x + \epsilon)^p = (x + \epsilon x_1 + \cdots + \epsilon)^p \]

\[ = (x_0 + \epsilon(x_1 + 1) + O(\epsilon^2))^p \]

\[ = x_0^p \left( 1 + \epsilon \frac{(x_1 + 1)}{x_0} + O(\epsilon^2) \right)^p \]

Then you can expand since the latter term has the form \((1 + z)^p\).

4. [25pts] Consider the perturbed first order initial value problem:

\[ \frac{dy}{dx} + y^2 = \epsilon \left( \frac{1}{y} - x \right), \quad y(0) = 1 \]

where \( 0 < \epsilon \ll 1 \). Find \( y_0(x) \) and \( y_1(x) \) in the assumed expansion of the solution \( y \):

\[ y(x, \epsilon) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2) \]

Here the leading \( O(1) \) problem for \( y_0(x) \) is a separable and

\[ \frac{1}{y} = \frac{1}{y_0 + \epsilon y_1 + O(\epsilon^2)} = \frac{1}{y_0} + O(\epsilon) \]