Math 450 (2017) – Midterm 1 (Take home)

Due: October 27, 2017. NAME: ______________________________

1. [30pts] A fluid of density ρ exerts a (drag) force F on a cylinder of diameter D as it flows around it. The fluid has viscosity μ and the fluid velocity far from the cylinder is v. Assume the physical law

$$f(\rho, F, D, \mu, v) = 0$$

and then find all the dimensionless π of the form

$$\pi = \rho^{\alpha_1} F^{\alpha_2} D^{\alpha_3} \mu^{\alpha_4} v^{\alpha_5}$$

Note that the units of viscosity are $[\mu] = ML^{-1}T^{-1}$.

2. [20pts] In quantum mechanics the wave function $\psi(X)$ of a particle of mass m is a solution of Schrödinger's equation. For the quantum harmonic oscillator problem, the (time-independent) Schrödinger equation is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dX^2} + \frac{1}{2}m\omega^2 X^2 \psi = E\psi$$

where X is position, ω is the frequency of the oscillator (potential well), $\hbar$ is Planck's constant and E is the energy of the system. The wave function ψ depends on position X but since it is a probability density function (used to determine the location of the particle) it is dimensionless and hence $[\psi] = 1$. Note here that $[\omega] = T^{-1}$ where T is time.

i) Determine the units of $\hbar$ in terms of M, T, L

ii) Nondimensionalize the problem so it has only one dimensionless quantity, i.e.

$$-\frac{d^2\psi}{dx^2} + (x^2 - \mathcal{E})\psi = 0$$

Note: Since ψ is already dimensionless, you only need to rescale the variable X:

$$x = \frac{X}{X^*},$$

Determine the parameter X^* which will yield the dimensionless equation in (1). Clearly express X^* and the dimensionless parameter $\mathcal{E}$ in terms of the original dimensional parameters $m, \omega, \hbar$ and E.

3. [25pts] The equation

$$f(x,\epsilon) = x - \frac{1}{(x+\epsilon)^3} = 0 \quad , \quad 0 < \epsilon \ll 1$$

has two real roots $\bar{x}_{\pm}(\epsilon)$. Assume

$$\bar{x}(\epsilon) = x_0 + x_1\epsilon + O(\epsilon^2)$$

and then determine x_0 and x_1 for both roots. If you use the Binomial theorem note

$$\begin{aligned}
(x+\epsilon)^p &= (x_0 + \epsilon x_1 + \cdots + \epsilon)^p \\
&= (x_0 + \epsilon(x_1+1) + O(\epsilon^2))^p \\
&= x_0^p \left(1 + \epsilon\frac{(x_1+1)}{x_0} + O(\epsilon^2)\right)^p
\end{aligned}$$

Then you can expand since the latter term has the form $(1+z)^p$.

4. [25pts] Consider the perturbed <u>first order</u> initial value problem:

$$\frac{dy}{dx} + y^2 = \epsilon\left(\frac{1}{y} - x\right) \quad , \quad y(0) = 1$$

where $0 < \epsilon \ll 1$. Find $y_0(x)$ and $y_1(x)$ in the assumed expansion of the solution y:

$$y(x,\epsilon) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2)$$

Here the leading $O(1)$ problem for $y_0(x)$ is a separable and

$$\frac{1}{y} = \frac{1}{y_0 + \epsilon y_1 + O(\epsilon^2)} = \frac{1}{y_0} + O(\epsilon)$$