Math 450 (2023) – Midterm 1 (Take home)

Due: Friday, October 20, 2023. NAME: _____

Instructions You may use your notes, posted class notes, the textbook and any differential equations textbook. You may not use software. Staple this sheet to your work when you turn it in (in class) on Friday 10/20/23. You must work alone but may ask me a clarifying question.

1. [30pts] A fluid of density ρ exerts a (drag) force F on a cylinder of diameter D as it flows around it. The fluid has viscosity μ and the fluid velocity far from the cylinder is v. Assume the physical law

$$f(\rho, F, D, \mu, v) = 0$$

and then find all the dimensionless π of the form

$$\pi = \rho^{\alpha_1} F^{\alpha_2} D^{\alpha_3} \mu^{\alpha_4} v^{\alpha_5}$$

Note that the units of viscosity are $[\mu] = ML^{-1}T^{-1}$.

2. [20pts] In quantum mechanics the wave function $\psi(X)$ of a particle of mass m is a solution of Schrödinger's equation. For the quantum harmonic oscillator problem, the (time-independent) Schrödinger equation is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dX^2} + \frac{1}{2}m\omega^2 X^2 \psi = E\psi$$

where X is position, ω is the frequency of the oscillator (potential well), $\hbar$ is Planck's constant and E is the energy of the system. The wave function ψ depends on position X but since it is a probability density function (used to determine the location of the particle) <u>it has no units</u>. Note here that $[\omega] = T^{-1}$ where T is time.

i) Determine the units of $\hbar$ in terms of M, T, L

ii) Nondimensionalize the problem so it has only one dimensionless parameter $\mathcal{E}$, i.e.,

$$-\frac{d^2\psi}{dx^2} + (x^2 - \mathcal{E})\psi = 0 \qquad , \quad x = \frac{X}{X^*}$$

Specifically find X^* and $\mathcal{E}$ in terms of the dimensional parameters $\hbar, m, \omega, E$.

3. [25pts] The equation

$$f(x,\epsilon) = x - \frac{1}{(x+\epsilon)^3} = 0 \quad , \quad 0 < \epsilon \ll 1$$

has two real <u>regular</u> roots $\bar{x}_\pm(\epsilon)$. Assume

$$\bar{x}(\epsilon) = x_0 + x_1\epsilon + O(\epsilon^2)$$

and then determine x_0 and x_1 for both roots. If you use the Binomial theorem note

$$\begin{aligned}
(x+\epsilon)^p &= (x_0 + \epsilon x_1 + \cdots + \epsilon)^p \\
&= (x_0 + \epsilon(x_1+1) + O(\epsilon^2))^p \\
&= x_0^p \left(1 + \epsilon\frac{(x_1+1)}{x_0} + O(\epsilon^2)\right)^p \\
&= x_0^p\,(1+z)^p
\end{aligned}$$

Then you can expand the latter term:

$$(1+z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + O(z^3)$$

for small z.

4. [25pts] Consider the perturbed first order initial value problem:

$$\frac{dy}{dx} + y^2 = \epsilon\left(\frac{1}{y} - x\right) \quad , \quad y(0) = 1$$

where $0 < \epsilon \ll 1$. Find $y_0(x)$ and $y_1(x)$ in the assumed expansion of the solution y:

$$y(x,\epsilon) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2)$$

Here the leading $O(1)$ problem for $y_0(x)$ is a <u>separable</u> first order equation. The $O(\epsilon)$ equation for y_1 is a linear first order equation hence can be solved using an integrating factor.