Math 451 (2024) – Midterm

Due: Friday, March 22, 2024. (in class) NAME: ________________________

Instructions You may use your notes, posted class notes, the textbook and any differential equations textbook. You may use software. Staple this sheet to your work when you turn it in (in class) on 3/22/24. You must work alone but may ask me clarifying questions.

1. [25pts] Find **all** the eigenvalues λ_n and associated **normalized** eigenfunctions $\phi_n(x)$ satisfying

$$L\phi_n = \lambda_n \phi_n \qquad , \qquad \phi_n \in D$$

where $Lu = u''$ and $D = \{u \in C^2[0,1] : u'(0) = 0, u'(1) = 0\}$. Here $n = 0, 1, 2, ...$

2. [20pts] A Sturm Liouville Problem has eigenfunctions $\phi_n(x) = \cos nx$, $n \geq 0$, $x \in [0, \pi]$.

a) [15] Find the normalized eigenfunctions and the Fourier coefficients c_n of $f(x) = x$.

$$\begin{aligned}
\hat{\phi}_0(x) &= \phi_0(x)/ \parallel \phi_0 \parallel && (1)\\
\hat{\phi}_n(x) &= \phi_n(x)/ \parallel \phi_n \parallel \quad , \quad n \geq 1 && (2)\\
f(x) &= \sum_{n=0}^{\infty} c_n \phi_n(x) \qquad , \qquad c_n = \left\langle f, \hat{\phi}_n \right\rangle && (3)
\end{aligned}$$

b) [5] Parseval's Theorem for these eigenfunctions and $f(x)$ is

$$\frac{\pi^2}{2} = \parallel f \parallel^2 = \sum_{n=0} c_n^2 = \cdots$$

Use this to find a formula for the sum

$$S = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$$

You need to use $\cos n\pi = (-1)^n$.

3. [30pts] Find the Green's function $g(x, \zeta)$ for the following (SLP):

$$\begin{aligned}
-\frac{d^2u}{dx^2} &= f(x) \qquad , \qquad x \in (0,1)\\
u(0) + u'(0) &= 0\\
u(1) - u'(1) &= 0
\end{aligned}$$

4. [10 pts] Use the properties of distributions to show

a) $x\delta'(x) = -\delta(x)$

b) $\alpha(x)\delta'(x) = -\alpha'(0)\delta(x) + \alpha(0)\delta'(x)$ where $\alpha \in C_0^\infty(\mathbb{R})$.

Look at pgs 10-11 of the posted notes on Distributions.

5. [15pts] Let $H(x)$ be the Heaviside step function where $H'(x) = \delta(x)$. Show the following product rule applies in the distributional sense

$$\frac{d}{dx}(xH(x)) = H(x) + xH'(x)$$

Specifically, fill in the missing steps for an arbitrary test function ϕ:

$$\left\langle (xH(x))', \phi \right\rangle = -\left\langle xH(x), \phi' \right\rangle = \cdots = \left\langle H(x) + xH'(x), \phi \right\rangle$$

Also use the distribution properties at pgs 10-11 of the posted notes on Distributions.