Math 450 (2023) – Homework 1

Due: Wednesday, September 13, 2023. NAME: ________________________

Convention Directions: In most of the problems below you derive a dimension matrix A whose nullspace defines the dimensionless groups. Set your matrix up so the i^{th} column corresponds to α_i in each problem. Also, order the rows so that the first row of A corresponds to L (length), the second row corresponds to M (mass), the third row corresponds to T (time) and lastly K (degrees Kelvin).

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1. [10pts] Deep water oceanic waves of wavelength λ are observed to travel at a constant speed v. Because these waves are deep they are thought to be primarily a gravitational phenomena. Assume that λ, v and the gravitational constant g are related:

$$f(\lambda, v, g) = 0$$

a) Find all the dimensionless parameters π associated with the problem where

$$\pi = \lambda^{\alpha_1} v^{\alpha_2} g^{\alpha_3}$$

b) Use the π-theorem to derive a formula for v in terms of the other dimensional quantities. What power of λ is v proportional to?

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2. [15pts] Bubbles rise in fluids. If you've ever seen videos of scuba divers you should have noticed large bubbles rise faster. Thus, the bubble speed v depends on the bubble volume V. The density ρ_0 of the gas in the bubble, the fluid density ρ and gravity g all affect the bubble velocity. Assume all the aforementioned dimensional quantities are related:

$$f(\rho, \rho_0, V, v, g) = 0$$

a) Find all the dimensionless parameters π associated with the problem where

$$\pi = \rho^{\alpha_1} \rho_0^{\alpha_2} V^{\alpha_3} v^{\alpha_4} g^{\alpha_5}$$

b) Use the π-theorem to derive a formula for v in terms of the other dimensional quantities.

c) If a bubble of volume V rises at $4cm/sec$ then how fast does a bubble of four times the volume rise?

3. [10pts] Boltzman studied the electromagnetic energy radiated by substances at different temperatures τ. He believed such radiated energy could only be explained with both quantum mechanics and thermodynamics so should involve the key parameters: Planck's constant $\hbar$ and Boltzmann's constant k where $[k] = joules/K$ where $K =^\circ Kelvin$. Planck's constant has units of joules $\times$ time and the radiation travels at the speed of light c. The energy <u>density</u> $\mathcal{E}$ $(joules/L^3)$ he thought was related to the former parameters. Assume

$$f(\mathcal{E}, \tau, \hbar, k, c) = 0 \quad ,$$

for some function f and then use dimensional analysis to find a formula for $\mathcal{E}$ in terms of dimensional quantities. What power of τ is $\mathcal{E}$ proportional to?

4. [10pts] The population of bacteria in a container of volume V are a fed soluble nutrient at a constant rate F (volume per time). A set of differential equations which models the bacteria concentration $N(t)$ and nutrient concentration $C(t)$ is:

$$\begin{aligned}
\frac{dN}{dt} &= \left(\frac{K_{max}C}{K_n + C}\right) N - \frac{FN}{V} \\
\frac{dC}{dt} &= -\alpha \left(\frac{K_{max}C}{K_n + C}\right) N - \frac{FC}{V} + \frac{FC_0}{V}
\end{aligned}$$

Here $[N] = ML^{-3}, [C] = ML^{-3}$ and the parameters $K_{max}, K_n, F, V, \alpha, C_0$ are constant. We won't worry about why these equations are appropriate but seek to simplify their analysis by first nondimensionalizing them.

a) Determine the dimensions of $K_{max}, K_n, F, V, \alpha, C_0$ in terms of the fundamental units M, L, T.

b) Define dimensionless variables

$$n = \frac{N}{N^*} \quad , \quad c = \frac{C}{C^*} \quad , \quad \tau = \frac{t}{t^*}$$

Find N^*, C^* and t^* such that the nondimensionalized system is

$$\begin{aligned}
\frac{dn}{d\tau} &= \alpha_1 \left(\frac{c}{1+c}\right) n - n \\
\frac{dc}{d\tau} &= -\left(\frac{c}{1+c}\right) n - c + \alpha_2
\end{aligned}$$

Write out formulae defining α_1, α_2. Note that the original system had 6 parameters versus 2 for the dimensionless system!