Math 450 (2017) – Homework 3

Due: Wednesday, October 18, 2017. NAME: ____________________

1. [10pts] Let $x = \bar{x}(\epsilon)$ be the root of $f(x, \epsilon) = 0$ and assume

$$\bar{x}(\epsilon) = x_0 + x_1\epsilon + x_2\epsilon^2 + O(\epsilon^3)$$

This assumption can be verified with the implicit function theorem. Find x_0, x_1 and x_2 for the following functions:

a) $f(x, \epsilon) = x^3 + 3\epsilon x^2 + 4\epsilon^2 - 8$

b) $f(x, \epsilon) = exp\left(\frac{1}{2}x\right) - \frac{2}{\sqrt{2+\epsilon x}}$

For the latter you may use the binomial expansion:

$$(1 + z)^{-1/2} = 1 - \frac{1}{2}z + \frac{3}{8}z^2 + O(z^3) \qquad , \qquad z = \frac{\epsilon x}{2}$$

Organize your work by summarizing the $O(1), O(\epsilon)$ and $O(\epsilon^2)$ problems.

2. [5pts] One can find regular expansions for solutions to coupled algebraic equations. Consider the coupled system

$$\begin{aligned} x^2 - x &= \epsilon(x + 6y) \\ x^2 - y^2 &= \epsilon x^2 \end{aligned}$$

Assume the exact solutions $(\bar{x}(\epsilon), \bar{y}(\epsilon))$ can be expanded as follows:

$$\bar{x}(\epsilon) = x_0 + x_1\epsilon + O(\epsilon^2)$$
$$\bar{y}(\epsilon) = y_0 + y_1\epsilon + O(\epsilon^2)$$

Find x_0, x_1, y_0 and y_1 for the two nonzero roots.

3. [8pts] Consider the initial value problem:

$$y'' + 4y = \ln(1 + \epsilon y) \quad , \quad y(0) = 0 \quad , \quad y'(0) = 2$$

where $0 < \epsilon \ll 1$. Find $y_0(t)$ and $y_1(t)$ in the assumed expansion of the solution y:

$$y(t, \epsilon) = y_0(t) + \epsilon y_1(t) + O(\epsilon^2)$$

4. [7pts] Regular perturbation techniques can be applied to approximate the solution of two-point Boundary Value Problems (BVP). Let $y(x)$ be the solution of the nonlinear BVP

$$y'' - \frac{18}{y'} = 21\epsilon(y-1) \quad , \quad y(0) = 1 \quad , \quad y(1) = 5$$

where $0 < \epsilon \ll 1$. Notice that the values of $y(x, \epsilon)$ are specified at the boundaries $x = 0$ and $x = 1$. We shall assume y has the expansion:

$$y(x, \epsilon) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2)$$

a) Clearly define and solve the $O(1)$ problem for $y_0(x)$. In particular show that

$$y_0(x) = 4x^{3/2} + 1$$

b) Derive and then solve the $O(\epsilon)$ problem for $y_1(x)$. Note that constants are solutions to the homogeneous problem for y_1.