Math 450 (2017) – Homework 3

Due: Wednesday, October 18, 2017.

NAME: ____________________________

1. [10pts] Let \( x = \bar{x}(\epsilon) \) be the root of \( f(x, \epsilon) = 0 \) and assume

\[
\bar{x}(\epsilon) = x_0 + x_1 \epsilon + x_2 \epsilon^2 + O(\epsilon^3)
\]

This assumption can be verified with the implicit function theorem. Find \( x_0, x_1 \) and \( x_2 \) for the following functions:

a) \( f(x, \epsilon) = x^3 + 3\epsilon x^2 + 4\epsilon^2 - 8 \)

b) \( f(x, \epsilon) = \exp\left(\frac{1}{2}x\right) - \frac{2}{\sqrt{x+\epsilon}} \)

For the latter you may use the binomial expansion:

\[
(1 + z)^{-1/2} = 1 - \frac{1}{2}z + \frac{3}{8}z^2 + O(z^3) \quad , \quad z = \frac{\epsilon x}{2}
\]

Organize your work by summarizing the \( O(1), O(\epsilon) \) and \( O(\epsilon^2) \) problems.

2. [5pts] One can find regular expansions for solutions to coupled algebraic equations. Consider the coupled system

\[
\begin{align*}
x^2 - x &= \epsilon(x + 6y) \\
x^2 - y^2 &= \epsilon x^2
\end{align*}
\]

Assume the exact solutions \((\bar{x}(\epsilon), \bar{y}(\epsilon))\) can be expanded as follows:

\[
\begin{align*}
\bar{x}(\epsilon) &= x_0 + x_1 \epsilon + O(\epsilon^2) \\
\bar{y}(\epsilon) &= y_0 + y_1 \epsilon + O(\epsilon^2)
\end{align*}
\]

Find \( x_0, x_1, y_0 \) and \( y_1 \) for the two nonzero roots.

3. [8pts] Consider the initial value problem:

\[
y'' + 4y = \ln(1 + \epsilon y) \quad , \quad y(0) = 0 \quad , \quad y'(0) = 2
\]

where \( 0 < \epsilon \ll 1 \). Find \( y_0(t) \) and \( y_1(t) \) in the assumed expansion of the solution \( y \):

\[
y(t, \epsilon) = y_0(t) + \epsilon y_1(t) + O(\epsilon^2)
\]
4. [7pts] Regular perturbation techniques can be applied to approximate the solution of two-point Boundary Value Problems (BVP). Let \( y(x) \) be the solution of the nonlinear BVP

\[
y'' - \frac{18}{y'} = 21\epsilon(y - 1) \quad , \quad y(0) = 1 \quad , \quad y(1) = 5
\]

where \( 0 < \epsilon \ll 1 \). Notice that the values of \( y(x, \epsilon) \) are specified at the boundaries \( x = 0 \) and \( x = 1 \). We shall assume \( y \) has the expansion:

\[
y(x, \epsilon) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2)
\]

a) Clearly define and solve the \( O(1) \) problem for \( y_0(x) \). In particular show that

\[
y_0(x) = 4x^{3/2} + 1
\]

b) Derive and then solve the \( O(\epsilon) \) problem for \( y_1(x) \). Note that constants are solutions to the homogeneous problem for \( y_1 \).