Math 450 (2017) – Homework 4

Due: Thur, November 9, 2017. NAME: ____________________
Turn it in during class Wed 11/8 or
slide under my door (Wil 2-236) before noon on Thur, 11/9

1. [10 pts] Let $y(t, \epsilon)$ be the solution of the initial value problem

$$\begin{aligned} y'' + y &= \epsilon\, y(y')^2 \quad , \quad 0 < \epsilon \ll 1 \\ y(0) &= 0 \quad , \quad y'(0) = 1 \end{aligned}$$

where $(\ )'$ denotes differentiation with respect to t. Assume

$$\begin{aligned} y(t, \epsilon) &= y_0(\tau) + \epsilon y_1(\tau) + O(\epsilon^2) \\ \tau &= \omega(\epsilon) \equiv 1 + \omega_1 \epsilon + O(\epsilon^2) \end{aligned}$$

where $y_k(\tau)$ are $2\pi-$periodic in τ for appropriate choices of ω_k for $k \geq 1$. Use Poincare Lindstedt's method to find $y_0(\tau)$ and the corrected period of the oscillation, i.e., T_0 and T_1 in the exact period (in the original time t):

$$T(\epsilon) = T_0 + \epsilon T_1 + O(\epsilon^2)$$

You will need to look up appropriate trigonometric identities to complete the problem.

2. [9 pts] <u>Prove</u> that as $\epsilon \to 0^+$ the following are true:

$$\begin{aligned} e^{-1/\epsilon} &\ll \epsilon^n \qquad , \quad \forall n > 0 \\ \int_0^\epsilon f(x)\, dx &= O(\epsilon) \\ log(\epsilon) &\ll \frac{1}{1 - cos(\epsilon)} \end{aligned}$$

For the first, consider the log of the ratio $e^{-1/\epsilon}/\epsilon^n$ to make the conclusion. The second can be proved using the Fundamental Theorem of Calculus or L'Hospital's rule and the third can be shown using L'Hospital's rule (though there is a simpler way).

3. [6pts] An asymptotic sequence $\{\phi_n(\epsilon)\}$ is defined by $\phi_n(\epsilon) = sin^n\epsilon$ for $n \geq 0$ noting $\phi_0 = 1$. Find constants a_0, a_1, a_2 and a_3 such that

$$f(\epsilon) \equiv \sqrt{1 - 4\epsilon} \sim a_0\phi_0(\epsilon) + a_1\phi_1(\epsilon) + a_2\phi_2(\epsilon) + a_3\phi_3(\epsilon) + O(\phi_4) \quad as \quad \epsilon \to 0$$

Hint: expand both sides in powers of ϵ, i.e. $\sin\epsilon = \epsilon - \frac{1}{3!}\epsilon^3 + \cdots$, the same for $\sqrt{1-4\epsilon}$, $\sin^2\epsilon$ and equate coefficients in ϵ^n on both sides.

4. [10 pts] Consider the equation

$$f(x,\epsilon) = \epsilon x^2 - \sqrt{x} + 1 = 0$$

Using calculus one can prove that there are exactly two positive roots to the above equation. If you plot ϵx^2 and $\sqrt{x} - 1$ you can quickly see that for ϵ small, one root is $O(1)$ and the other is singular in ϵ.

a) Compute x_0, x_1 in the regular expansion

$$\bar{x}_-(\epsilon) = x_0 + x_1\epsilon + O(\epsilon^2)$$

b) For the singular root, determine X_0, X_1 and α in the expansion

$$\bar{x}_+(\epsilon) = \frac{1}{\epsilon^\alpha}\left(X_0 + \delta X_1 + O(\delta^2)\right) \quad , \quad \alpha > 0$$

for an appropriate function $\delta(\epsilon) \ll 1$.

5. [10 pts] Find the leading inner and outer solutions $y_0(x)$ and $Y_0(X)$ of the boundary value problem

$$\begin{aligned} \epsilon y'' + y' + y^2 &= 0 \quad , \quad x \in (0,1) \\ y(0) &= \frac{1}{4} \quad , \\ y(1) &= \frac{1}{2} \quad , \end{aligned}$$

and then a uniformly valid approximation $y_u(x,\epsilon)$.