Math 450 (2023) – Homework 4

Due: Friday, November 3, 2023. NAME: ______________________

Instructions: You may use software to help solve differential equations and plotting solutions.

1. [10 pts] Prove that as $\epsilon \to 0^+$ the following are true:

$$
\begin{aligned}
e^{-1/\epsilon} &\ll \epsilon^n \qquad , \quad \forall n > 0 \\
\int_0^\epsilon f(x)\, dx &= O(\epsilon) \\
log(\epsilon) &\ll \frac{1}{1 - cos(\epsilon)}
\end{aligned}
$$

For the first, consider the log of the ratio to make the conclusion. The middle can be proved using the Fundamental Theorem of Calculus and the last can be shown using L'Hospital's rule (though there is a simpler way).

2. [20 pts] For each of the following algebraic equations find the two term expansion of the singular root(s).

$$f(x, \epsilon) = \epsilon x^2 - \sqrt{x} + 1 = 0 \tag{1}$$

$$f(x, \epsilon) = \epsilon x^4 + x^3 + x + 1 = 0 \tag{2}$$

Specifically, for the singular root X determine X_0, X_1, α and $\delta(\epsilon) \ll 1$ in the expansion

$$X = \frac{1}{\epsilon^\alpha}\left(X_0 + \delta X_1 + O(\delta^2)\right) \quad , \quad \alpha > 0$$

3. [10 pts] Let $y(x, \epsilon)$ be the solution of the nonlinear boundary value problem:

$$
\begin{aligned}
\epsilon y'' + (1 + 2x)y' - 2y &= 0 \quad , \quad 0 < x < 1 \\
y(0) &= A \quad , \quad y(1) = 3
\end{aligned}
$$

where $(\ )'$ denotes differentiation with respect to x and $0 < \epsilon \ll 1$.

a) Show that when $A = 1$ the leading order outer solution $y_0(x)$ satisfies both boundary conditions. This is an example of a case when there is no layer.

b) For the case $A = 2$, find the uniformly valid solution $y_u(x, \epsilon)$ having a boundary layer at $x = 0$. Sketch your solution for $\epsilon = 0.01$.

4. [10 pts] Easy Nonlinear BVP:

$$\begin{aligned} \epsilon y'' + y' - e^y &= 0 \quad , \quad 0 < x < 1 \\ y(0) &= 0 \quad , \quad y(1) = 0 \end{aligned}$$

where $(\ )'$ denotes differentiation with respect to x and $0 < \epsilon \ll 1$. Find the uniformly valid approximation $y_u(x, \epsilon)$ having a layer at $x = 0$. In the layer, $e^y = O(1)$ so is not involved in the dominant balance. When done, sketch your solution for $\epsilon = 0.01$.