Math 450 (2013) – Homework 5

Due: December 2, 2013.

1. [7 pts] Nonlinear BVP:

$$\begin{aligned} \epsilon y'' + yy' - y\sec(y) &= 0 \quad , \quad 0 < x < 1 \\ y(0) &= 0 \quad , \quad y\left(\frac{1}{2}\right) = \frac{\pi}{2} \end{aligned}$$

where $(\ )'$ denotes differentiation with respect to x and $0 < \epsilon \ll 1$. Find the uniformly valid approximation $y_u(x, \epsilon)$ having a layer at $x = 0$. In the layer, $y\sec(y) = O(1)$ so is not involved in the dominant balance. Also, the leading outer solution is differentiable on $(0, \frac{1}{2})$ but not at $x = \frac{1}{2}$ (minor detail). When done, sketch your solution. A good value to see the layer well is $\epsilon = 0.005$.

2. [8 pts] For the functionals $J(y)$ defined in a) and b)

i) Define the set of admissible variations $\mathcal{A}^*$ so that $y + h \in \mathcal{A}$ whenever $h \in \mathcal{A}^*$.

ii) Derive a formula for $\delta J(y, h)$ for general y, h.

iii) Use your result in ii) to compute $\delta J(\bar{y}, h)$ for the specific $\bar{y}, h$ listed.

---

a) Let $J : \mathcal{A} \to \mathbb{R}$ where $\mathcal{A} = \{y \in C[0, \pi] : y'(0) = 1, y'(\pi) = -1\}$. For $\alpha \in [0, \pi]$ define

$$J(y) \equiv y(\alpha)y'(\alpha) \quad , \quad \bar{y}(x) = \sin x \quad , \quad h(x) = \cos x$$

Should get something that looks like the product rule for ii)

b) Let $J : \mathcal{A} \to \mathbb{R}$ where $\mathcal{A} = C^2[0, 1]$.

$$J(y) \equiv \int_0^1 \sqrt{3x + y(x)y'(x)}\, dx \quad , \quad \bar{y}(x) = x \quad , \quad h(x) = 2x - 1$$

3. [5 pts] Use the Euler Lagrange equations to find all possible extrema $\bar{y} \in \mathcal{A}$ of

$$\begin{aligned} J(y) &\equiv \int_0^\pi 4y'(x)^2 + 2y(x)y'(x) - y(x)^2\, dx \\ \mathcal{A} &= \{y :\ y \in C^2[0, \pi],\ y'(0) = A,\ y(\pi) = 0\} \end{aligned}$$

for $A = 0$ and $A \neq 0$. Ask yourself: are there extrema and how many?

4. [5 pts] Find the extrema of $J(y)$ defined by

$$\begin{aligned} J(y) &\equiv \int_0^{\pi/4} y'(x)^2\, cos^2 x\, dx \\ \mathcal{A} &= \{y :\ y \in C^1[0, \pi/4],\ y(0) = 1,\ y(\pi/4) = 2\ \} \end{aligned}$$