Math 450 (2023) – Homework 5

Due: Friday November 17, 2023. NAME: _____

1. [10 pts] For the functionals $J(y)$ defined in a) and b)

i) Define the set of admissible variations $\mathcal{A}^*$ so that $y + h \in \mathcal{A}$ whenever $h \in \mathcal{A}^*$.

ii) Derive a formula for $\delta J(y, h)$ for general y, h.

iii) Use your result in ii) to compute $\delta J(\bar{y}, h)$ for the specific $\bar{y}, h$ listed.

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a) Let $J : \mathcal{A} \to \mathbb{R}$ where $\mathcal{A} = \{y \in C[0, \pi] : y'(0) = 1, y'(\pi) = -1\}$. For $\alpha \in [0, \pi]$ define

$$J(y) \equiv y(\alpha)y'(\alpha) \quad , \quad \bar{y}(x) = \sin x \quad , \quad h(x) = \cos x$$

Should get something that looks like the product rule for ii)

b) Let $J : \mathcal{A} \to \mathbb{R}$ where $\mathcal{A} = C^2[0, 1]$.

$$J(y) \equiv \int_0^1 \sqrt{3x + y(x)y'(x)}\, dx \quad , \quad \bar{y}(x) = x \quad , \quad h(x) = 2x - 1$$

2. [5 pts] Use the Euler Lagrange equations to find the extrema $\bar{y}(x)$ of

$$J(y) \equiv \int_0^1 \; y(x) + log(1 + y'(x))\, dx$$

over the admissible set

$$\mathcal{A} = \{y : \; y \in C^2[0, 1], \; y(0) = 0, \; y(1) = 0\}$$

3. [5 pts] Use the Euler Lagrange equations to find all possible extrema $\bar{y} \in \mathcal{A}$ of

$$\begin{aligned} J(y) &\equiv \int_0^\pi \; 4y'(x)^2 + 2y(x)y'(x) - y(x)^2 \, dx \\ \mathcal{A} &= \{y : \; y \in C^2[0, \pi], \; y'(0) = A, \; y(\pi) = 0\} \end{aligned}$$

for $A = 1$ and $A = 0$. Ask yourself: are there extrema and how many?

4. [5 pts] Find all the natural boundary conditions associated with extremizing $J(y)$ over $\mathcal{A}$ for the following functionals and admissible sets.

$$\begin{aligned} J(y) &= y(1)^2 + \int_0^1 \left(y^2 - xyy'^2\right)\, dx \\ \mathcal{A} &= \{y : \; y \in C^2[0,1], y(0) = 1\} \end{aligned}$$

Do not find the extrema. Just derive all the natural boundary conditions. It is easiest if one derives the conditions using a general Lagrangian. For example, start with

$$J(y) = y(1) + \int_0^1 L(x, y, y')\, dx$$

for an arbitrary Lagrangian L.