Math 451 (2024) – Homework 2

(THROUGHOUT USE INTEGRAL TABLES AND ODE SOLVERS AS NEEDED)

Due: Friday, Feb. 9, 2024. NAME: ____________________

1. [5 pts] Define $f_\lambda(x) \equiv \sin(\lambda x) \in L^2[0, \pi]$ for $\lambda \in \mathbb{R}$. Here λ need not be an integer.

a) Compute $F(\lambda) \equiv \| f_\lambda \|$ for all $\lambda \neq 0$ and show F is constant on the integers.

b) Compute $G(\lambda, \mu) \equiv < f_\lambda, f_\mu >$ for $\lambda \neq \pm\mu$

c) Show that for nonzero λ, μ the functions f_λ and f_μ are orthogonal only if

$$h(\lambda) = h(\mu) \qquad where \quad h(z) = \frac{\tan(\pi z)}{z}$$

Aside from getting you to compute norms and inner products, the result in c) can be used to show that if one fixes μ at some value then there exists $\lambda_n \neq n$ such that $\{\phi_n(x)\} = \{\sin(\lambda_n x)\}_{n\geq 1}$ is an orthogonal set much like $\{\sin(nx)\}_{n\geq 1}$ was shown to be in class.

2. [6 pts] The Legendre polynomials are a sequence of n^{th} degree polynomials $\{P_n(x)\}_{n\geq 1}$ defined by:

$$P_0(x) = 1 \tag{1}$$

$$P_1(x) = x \tag{2}$$

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \qquad n = 1, 2, \ldots \tag{3}$$

a) Use the recursion formula above to write out $P_2(x), P_3(x)$ and $P_4(x)$.

b) It can be shown that $\{P_n(x)\}$ are mutually orthogonal in $L^2[-1, 1]$. Use this fact to determine a_n in the following expansion

$$1 + x + x^2 = \sum_{n=0}^{\infty} a_n \; P_n(x)$$

You may use the fact that $a_n = 0, \forall n \geq 3$ (Why?) and $a_n = \frac{<f, P_n>}{<P_n, P_n>}$.

3. [6 pts] For every function $f(x) \in L^2[0, \pi]$ there are c_n such that

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(nx)$$

a) Compute c_n for $f(x) = sin(\beta x)$ where β is not an integer.

b) Use Parseval's identity and the result in a) to determine a formula for S where

$$S = \sum_{n=1}^{\infty} \frac{n^2}{(\beta^2 - n^2)^2}$$

4. [18 pts] Below are three regular Sturm-Liouville eigenvalue Problems (SLP)

$$\begin{array}{llll} (I) & y'' + \lambda y = 0 & y(0) = 0 & y'(1) = 0 \\ (II) & y'' + \lambda y = 0 & y(0) + y'(0) = 0 & y(1) = 0 \\ (III) & \frac{d}{dx}\left((2+x)^2 \frac{dy}{dx}\right) + \lambda y = 0 & y(-1) = 0 & y(1) = 0 \end{array} \tag{4}$$

Find all eigenvalues λ_n and associated eigenfunctions $y_n(x)$ for each of (I)-(III) above. When possible find an explicit formula for λ_n as in $\lambda_n = n^2$. If you can not find an explicit formula, λ_n will be roots of some function $f(z)$ as in $f(\lambda_n) = 0$. In those cases state what $f(z)$ is.

Remarks: For (II) you may want to write the general solution in the form:

$$y(x) = c_1 \sin(\mu(x-1)) + c_2 \cos(\mu(x-1))$$

Also, problem (III) is Cauchy-Euler in $z = x + 2$, i.e. $y(x) = (x+2)^r$ results in an equation for r in terms of λ. Take special note of the $\lambda > \frac{1}{4}$ case.