Math 451 (2024) – Homework 6

Due: Friday, January 26, 2024. NAME: ______________________

1. [10 pts] Find the extrema of

$$J(y) \equiv \int_0^1 xy(x)\ dx$$

over

$$\mathcal{A} = \left\{ y \in C^2[0,1] : y(0) = 0, y(1) = 0 \right\}$$

subject to the constraint

$$K(y) \equiv \int_0^1 y'(x)^2\ dx = 1$$

see: Notes 3g, pgs 6-10.

2. [5 pts] Consider the motion of a particle in the (x, y)-plane with Lagrangian

$$L = T - V = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) - V(r)$$

where $V(r)$ is the potential energy and T is the kinetic energy. Re-express the Lagrangian in polar coordinates (r, θ), i.e., $L = L(r, \dot{r}, \theta, \dot{\theta})$ (show your derivation of the kinetic energy term into polar coordinates). Then, write out the Euler-Lagrange equations. These are the equations of motion for planar motion of a particle under the influence of a radially symmetric force. Don't solve them. **see: Notes 3e, pgs 6.**

3. [5 pts] Let $\Gamma = (X(t), Y(t), Z(t))$ with $0 < t < 1$ be a geodesic on the the graph $z = y - 2x^2$. Assume that $y = y(x)$ is a function of x on Γ. Under this assumption show that for an appropriate Lagrangian $L(x, y, y')$ the length functional

$$J(y) = \int_0^1 \sqrt{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2}\ dt = \int_a^b L(x, y, y')\ dx$$

where $y' \equiv \frac{dy}{dx}$ and

$$L = \sqrt{1 + (y')^2 + (y' - 4x)^2}$$

Then the Euler-Lagrange equations are:

$$0 = \frac{d}{dx} L_{y'}$$

so that for some constant c

$$L_{y'} = c$$

Find the $y(x)$ that defines the geodesic satisfying the boundary conditions

$$y'(0) = 0 \qquad , \qquad y(1) = 1$$

see: Notes 3f(ii), much of it