Math 451 (2024) – Homework 8

(THROUGHOUT USE INTEGRAL TABLES AND ODE SOLVERS AS NEEDED)

Due: Wednesday February 28, 2024. NAME: ______________________

1. [10 pts] Find the Green's functions $g(x,\zeta)$ for the following problems:

$$
\begin{aligned}
-\frac{d^2u}{dx^2} &= f(x) \quad , \quad u(0) = u'(1) = 0 \\
-\frac{d}{dx}\left(x^2\frac{du}{dx}\right) + 2u &= f(x) \quad , \quad 2u(1) + u'(1) = u'(2) = 0
\end{aligned}
$$

2. [5 pts] Show that the eigenfunctions of

$$
\begin{aligned}
Lu &\equiv -u'' \qquad , \qquad u \in D \\
D &= \left\{u \in C^2[0,1] : u(0) = 0\ ,\ u(1) + u'(1) = 0\right\}
\end{aligned}
$$

are $u_n(x) = \sin\sqrt{\lambda_n}x$ for $n = 1, 2, \ldots$ where λ_n are roots of $F(\lambda) \equiv \tan\sqrt{\lambda} - \sqrt{\lambda}$. Using these eigenfunctions find the series representation of the Green's function $g(x,\zeta)$ for the following problem:

$$
Lu \equiv u'' = f(x) \quad , \quad u(0) = 0 \quad , \quad u(1) + u'(1) = 0
$$

The answer will involve the unknown values of λ_n.

3. [5 pts] Define the operator L and its domain by:

$$
\begin{aligned}
Lu &\equiv -(k(x)u')' \quad , \quad k(x) > 0 \\
D &\equiv \{u \in C^2[0,1] : u(0) = u(1) = 0\}
\end{aligned}
$$

Determine the inverse operator L^{-1}

$$
L^{-1}f = \int_0^1 g(x,\zeta)f(\zeta)d\zeta
$$

Write your answer in terms of the function

$$
K(x) \equiv \int_0^x \frac{1}{k(s)}\ ds
$$