Asymptotic Analysis

Defn (Little "oh") Let $f(\varepsilon)$ and $g(\varepsilon)$ be defined on some neighbourhood of $\varepsilon = 0$

(1) $\quad f(\varepsilon) = o(g(\varepsilon)) \quad$ as $\varepsilon \to 0$

if

$$\lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| = 0$$

Notation equivalent to (1) is

(2) $\quad f(\varepsilon) \ll g(\varepsilon) \quad$ as $\varepsilon \to 0$

Defn (Big "Oh") We write

$$f(\varepsilon) = O(g(\varepsilon)) \quad \text{as } \varepsilon \to 0$$

if there is a constant M such that

$$|f(\varepsilon)| \leq M\,|g(\varepsilon)|$$

for all ε in some neighbourhood of zero

Remarks: These definitions apply whether $f(\varepsilon)$ and $g(\varepsilon)$ are "small" or "large" as $\varepsilon \to 0$. For instance

$$\frac{1}{\varepsilon} \ll \frac{1}{\varepsilon^2}$$

where $f(\varepsilon) = \varepsilon^{-1}$, $g(\varepsilon) = \varepsilon^{-2}$ both grow as $\varepsilon \to 0^+$

EXAMPLE $\quad \varepsilon^3 \ll \varepsilon^2 \ll \varepsilon \ll 1 \ll \frac{1}{\varepsilon}$

are all obvious from the defn (1).
i.e.,

$$f(\varepsilon) = \varepsilon \qquad g(\varepsilon) = \frac{1}{\varepsilon}$$

Then

$$\left|\frac{f(\varepsilon)}{g(\varepsilon)}\right| = \varepsilon^2 \to 0 \quad \Rightarrow \quad \varepsilon \ll \frac{1}{\varepsilon}$$

EXAMPLE $\quad \varepsilon \ll \varepsilon \ln \varepsilon$

Since $\left|\frac{\varepsilon}{\varepsilon \ln \varepsilon}\right| = \frac{1}{|\ln \varepsilon|} \to 0$ as $\varepsilon \to 0$

EXAMPLE $\quad \varepsilon^n \ln \varepsilon \ll 1$ for all $n > 0$.

$$\lim_{\varepsilon \to 0^+} \varepsilon^n \ln \varepsilon = \lim_{\varepsilon \to 0^+} \frac{\ln \varepsilon}{\varepsilon^{-n}} \qquad \frac{\infty}{\infty} \text{ form}$$

L'Hopital's Rule

$$= \lim_{\varepsilon \to 0^+} \frac{\varepsilon^{-1}}{-n\varepsilon^{-n-1}}$$

$$= \lim_{\varepsilon \to 0^+} -\frac{1}{n}\varepsilon^n$$

$n > 0$

$$= 0$$

So for instance $\sqrt{\varepsilon} \ln \varepsilon \ll 1$ as $\varepsilon \to 0^+$

EXAMPLE $\quad \sin \varepsilon \ll 1$

since $\sin \varepsilon \to 0$ as $\varepsilon \to 0$

EXAMPLES INVOLVING BIG O

The statement $f = O(g)$ does not necessarily imply that f and g are "the same magnitude" but rather that their ratio is bounded. Some noteable examples:

$$\sin\left(\tfrac{1}{\varepsilon}\right) = O(1) \qquad \text{since } \left|\sin\left(\tfrac{1}{\varepsilon}\right)\right| \leq 2 = M$$

where here $\sin\left(\frac{1}{\varepsilon}\right)$ approaches nothing as $\varepsilon \to 0$. Also,

$$\varepsilon^2 = O(\varepsilon) \qquad \text{since } \left|\frac{\varepsilon^2}{\varepsilon}\right| = |\varepsilon| \leq 1 = M$$

for ε sufficiently small. This is true even though $\varepsilon^2 \ll \varepsilon$!, i.e. different "magnitudes"

A very useful theorem without proof

Theorem If the limit exists, then

$$\lim_{\varepsilon \to 0} \left|\frac{f(\varepsilon)}{g(\varepsilon)}\right| = L \leq 0 \quad \Rightarrow \quad f = O(g)$$

If $L > 0$ then $f = O(g)$ and $g = O(f)$

EXAMPLE $\sin\varepsilon = O(\varepsilon)$ since $\frac{\sin\varepsilon}{\varepsilon} \to 1$ as $\varepsilon \to 0$

EXAMPLE $2 + \varepsilon = O(1)$ since $\frac{2+\varepsilon}{1} \to 2$ as $\varepsilon \to 0$

EXAMPLE $\cot\varepsilon = O\left(\frac{1}{\varepsilon}\right)$ since

$$\lim_{\varepsilon \to 0} \frac{\cot\varepsilon}{\varepsilon^{-1}} = \lim_{\varepsilon \to 0} \frac{\cos\varepsilon \cdot \varepsilon}{\sin\varepsilon}$$

$$= \left(\lim_{\varepsilon \to 0} \cos\varepsilon\right)\left(\lim_{\varepsilon \to 0} \frac{\varepsilon}{\sin\varepsilon}\right)$$

$$= 1 \cdot 1 = 1$$

Uniformity (no x dependence)

Let $f(x,\varepsilon)$ and $g(x,\varepsilon)$ be defined for ε in the neighbourhood of zero and $x \in D$.

Then, the statements

$$f(x,\varepsilon) \ll g(x,\varepsilon) \qquad \varepsilon \to 0$$

$$f(x,\varepsilon) = O(g(x,\varepsilon)) \qquad \varepsilon \to 0$$

may hold true for certain fixed x values.

If they hold true for all $x \in D$ then they are said to hold uniformly on D.

EXAMPLE Let $f(x,\varepsilon) = \varepsilon^2 \sin x$, $g(x,\varepsilon) = \varepsilon x$

(1) $$\varepsilon^2 \sin x \ll \varepsilon x \qquad \text{as } \varepsilon \to 0$$

on any set D not containing $x=0$, since,

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^2 \sin x}{\varepsilon x} = \lim_{\varepsilon \to 0} \varepsilon \left(\frac{\sin x}{x} \right) = 0$$

so long as $x \neq 0$.

So for instance,

$$\varepsilon^2 \sin x \ll \varepsilon x \qquad \text{uniformly on } D = [\tfrac{1}{2}, 2]$$

<u>EXAMPLE</u> Consider

$$f(x,\varepsilon) = x + e^{-x/\varepsilon} \qquad g(x,\varepsilon) = x$$

where $D = [0,1]$. So long as $x \neq 0$

$$\lim_{\varepsilon \to 0^+} \frac{f(x,\varepsilon)}{g(x,\varepsilon)} = \lim_{\varepsilon \to 0^+} \frac{x + e^{-x/\varepsilon}}{x} = L = 1$$

so that for $x \neq 0$

$$(1) \qquad f(x,\varepsilon) = O(g(x,\varepsilon)) \qquad \text{as } \varepsilon \to 0^+$$

The statement (1) is uniformly true on $[\tfrac{1}{2}, 1]$ say but isn't on $[0,1]$. To see why, we ask does

$$f(0,\varepsilon) = O(g(0,\varepsilon)) \quad ?$$

The answer is no. If it were true there would be a constant M such that

$$|f(0,\varepsilon)| \leq M\,|g(0,\varepsilon)|$$

$$|0 + e^0| \leq M \cdot 0$$

$$1 \leq 0$$

a contradiction. Thus (1) depends on x and does not apply uniformly on $[0,1]$.

<u>Comment</u> The reasons for concern about "uniformity" are deep. See "Introduction to Perturbation Methods", M.H. Holmes for a good discussion

Asymptotic Sequences

Defn $\{\phi_n(\varepsilon)\}$ is an asymptotic sequence if

$$\phi_{n+1} \ll \phi_n \qquad \text{for all } n \geq 0$$

In words an asymptotic sequence is a sequence of asymptotically ordered ϕ_n.

EX $\quad \phi_n(\varepsilon) = \varepsilon^n \quad , \quad n \geq -2$

$$\frac{1}{\varepsilon^2} \gg \frac{1}{\varepsilon} \gg 1 \gg \varepsilon \gg \varepsilon^2 \gg \cdots$$

EX

$$\{\phi_n\} = \{\log\varepsilon,\ 1,\ \varepsilon\log\varepsilon,\ \varepsilon,\ \varepsilon^2(\log\varepsilon)^2,\ \varepsilon^2,\ \cdot\}$$

$$\{\phi_n\} = \{\ \phi_0,\ \phi_1,\ \phi_2,\ \phi_3,\ \phi_4,\ \phi_5\ \}$$

In particular note $\phi_4(\varepsilon) \ll \phi_3(\varepsilon)$, i.e.

$$\frac{\phi_4}{\phi_3} = \frac{\varepsilon^2(\log\varepsilon)^2}{\varepsilon} = \varepsilon(\log\varepsilon)^2 \to 0$$

where the latter vanishes (as $\varepsilon \to 0$) using L'Hopital's rule

$$\lim_{\varepsilon\to 0} \varepsilon(\log\varepsilon)^2 = \lim_{\varepsilon\to 0} \frac{(\log\varepsilon)^2}{\varepsilon^{-1}} \quad \text{L'Hopital}$$

$$= \lim_{\varepsilon\to 0} \frac{2\varepsilon^{-1}\log\varepsilon}{-\varepsilon^{-2}}$$

$$= \lim_{\varepsilon\to 0} -2\varepsilon\log\varepsilon$$

$$= 0 \qquad \text{by prev. ex (L'Hopital)}$$

Asymptotic approximations

Let $\{\phi_n(x,\varepsilon)\}$ be an asymptotic sequence of "guage functions".

The series

(1) $$S = \sum_{n=0}^{\infty} a_n \phi_n(x,\varepsilon) \qquad a_n \in \mathbb{R}$$

is said to be an asymptotic approximation of $y(x,\varepsilon)$ if

(2) $$y(x,\varepsilon) - \sum_{n=0}^{N} a_n \phi_n(x,\varepsilon) = o\left(\phi_{N+1}(x,\varepsilon)\right)$$

for all $N \geq 0$.

Remarks on error

Equation (2) is a statement about relative (not absolute) error.

EXAMPLE $\quad \phi_0(\varepsilon) = \frac{1}{\varepsilon^2}, \; \phi_1(\varepsilon) = \frac{1}{\varepsilon}, \; y = \frac{1}{\varepsilon^2} + \frac{1}{\sqrt{\varepsilon}}$

Note

$$\frac{y - \phi_0}{\phi_1} = \sqrt{\varepsilon} \to 0 \qquad \text{(relative error)}$$

so that $y - \phi_0 = o(\phi_1)$ but that

$$|y - \phi_0| = \frac{1}{\sqrt{\varepsilon}} \to \infty \qquad \text{absolute error}$$

Notation

Once the "guage functions" $\phi_n(x, \varepsilon)$ have been stipulated one can denote truncated asymptotic approximations and errors, i.e.,

$$y(x, \varepsilon) \underset{\text{"asymptotic to"}}{\sim} a_0 \phi_0(x, \varepsilon) + \underset{\text{with an error of this order.}}{o(\phi_1)}$$

which precisely means the following relative error is small

$$\frac{y(x, \varepsilon) - a_0 \phi_0(x, \varepsilon)}{\phi_1(x, \varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0$$

Nonuniqueness of representation

The same function $y(x, \varepsilon)$ can be represented with different asymptotic series:

$$\ln(1+\varepsilon) \sim \varepsilon + O(\varepsilon^2)$$

$$\ln(1+\varepsilon) \sim \sin\varepsilon + O(\varepsilon^2)$$

Divergent Series

Not discussed/proven but its possible for

$$y(x, \varepsilon) \sim S = \sum_{n=0}^{\infty} a_n \phi_n(x, \varepsilon)$$

even though S is a divergent series!