Asymptotic Analysis

**Defn (Little "oh")**  Let \( f(\varepsilon) \) and \( g(\varepsilon) \) be defined on some neighbourhood of \( \varepsilon = 0 \)

\[
1. \quad f(\varepsilon) = o(g(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0
\]

if

\[
\lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| = 0
\]

Notation equivalent to (1) is

\[
2. \quad f(\varepsilon) \ll g(\varepsilon) \quad \text{as} \quad \varepsilon \to 0
\]

**Defn (Big "Oh")** We write

\[
f(\varepsilon) = \bigO(g(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0
\]

if there is a constant \( M \) such that

\[
|f(\varepsilon)| \leq M |g(\varepsilon)|
\]

for all \( \varepsilon \) in some neighbourhood of zero

**Remarks:** These definitions apply whether \( f(\varepsilon) \) and \( g(\varepsilon) \) are "small" or "large" as \( \varepsilon \to 0 \). For instance

\[
\frac{1}{\varepsilon} \ll \frac{1}{\varepsilon^2}
\]

where \( f(\varepsilon) = \varepsilon^{-1}, \ g(\varepsilon) = \varepsilon^{-2} \)

both grow as \( \varepsilon \to 0^+ \)
EXAMPLE
\(\varepsilon^3 \ll \varepsilon^2 \ll \varepsilon \ll 1 \ll \frac{1}{\varepsilon}\)

are all obvious from the defn (1).

I.e.,

\[ f(\varepsilon) = \varepsilon \quad g(\varepsilon) = \frac{1}{\varepsilon} \]

Then

\[ \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| = \varepsilon^2 \to 0 \quad \Rightarrow \quad \varepsilon \ll \frac{1}{\varepsilon} \]

EXAMPLE
\(\varepsilon \ll \varepsilon \ln \varepsilon\)

Since 

\[ \left| \frac{\varepsilon}{\varepsilon \ln \varepsilon} \right| = \frac{1}{\ln \varepsilon} \to 0 \quad \text{as} \ \varepsilon \to 0 \]

EXAMPLE
\(\varepsilon^n \ln \varepsilon \ll 1 \quad \text{for all} \quad n > 0.\)

\[
\lim_{\varepsilon \to 0^+} \varepsilon^n \ln \varepsilon = \lim_{\varepsilon \to 0^+} \frac{\ln \varepsilon}{\varepsilon^{-n}} \xrightarrow{\text{form}} \infty
\]

\[ = \lim_{\varepsilon \to 0^+} \frac{\varepsilon^{-1}}{-n \varepsilon^{-n-1}} \quad \text{L'Hôpital's Rule} \]

\[ = \lim_{\varepsilon \to 0^+} -\frac{n}{n+1} \varepsilon^n \quad \text{if} \quad n > 0 \]

\[ = 0 \]

So for instance \(\sqrt{\varepsilon} \ln \varepsilon \ll 1 \quad \text{as} \ \varepsilon \to 0^+\)

EXAMPLE
\(\sin \varepsilon \ll 1\)

Since \(\sin \varepsilon \to 0 \quad \text{as} \ \varepsilon \to 0\)
EXEMPLARY INVOLVING BIG O

The statement \( f = O(g) \) does not necessarily imply that \( f \) and \( g \) are "the same magnitude" but rather that their ratio is bounded. Some notable examples:

\[
\sin \left( \frac{1}{\varepsilon} \right) = O(1) \quad \text{since} \quad |\sin \left( \frac{1}{\varepsilon} \right)| \leq 2 = M
\]

where here \( \sin \left( \frac{1}{\varepsilon} \right) \) approaches nothing as \( \varepsilon \to 0 \).

Also,

\[
\varepsilon^2 = O(\varepsilon) \quad \text{since} \quad \left| \frac{\varepsilon^2}{\varepsilon} \right| = |\varepsilon| \leq 1 = M
\]

for \( \varepsilon \) sufficiently small. This is true even though \( \varepsilon^2 \ll \varepsilon \), i.e., different "magnitudes".

A very useful theorem without proof:

<table>
<thead>
<tr>
<th>Theorem</th>
<th>If the limit exists, then</th>
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<tbody>
<tr>
<td>[ \lim_{\varepsilon \to 0} \left</td>
<td>\frac{f(\varepsilon)}{g(\varepsilon)} \right</td>
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If \( L > 0 \) then \( f = O(g) \) and \( g = O(f) \)

Example: \( \sin \varepsilon = O(\varepsilon) \) since \( \frac{\sin \varepsilon}{\varepsilon} \to 1 \) as \( \varepsilon \to 0 \)

Example: \( 2 + \varepsilon = O(1) \) since \( \frac{2 + \varepsilon}{1} \to 2 \) as \( \varepsilon \to 0 \)

Example: \( \cot \varepsilon = O(\frac{1}{\varepsilon}) \) since

\[
\lim_{\varepsilon \to 0} \cot \varepsilon = \lim_{\varepsilon \to 0} \frac{\cos \varepsilon \cdot \varepsilon}{\sin \varepsilon} = (\lim_{\varepsilon \to 0} \cos \varepsilon)(\lim_{\varepsilon \to 0} \frac{\varepsilon}{\sin \varepsilon}) = 1 \cdot 1 = 1
\]
Uniformity (no $x$ dependence)

Let $f(x, \varepsilon)$ and $g(x, \varepsilon)$ be defined for $\varepsilon$ in the neighbourhood of zero and $x \in D$.

Then, the statements

$$f(x, \varepsilon) \ll g(x, \varepsilon) \quad \varepsilon \to 0$$

$$f(x, \varepsilon) = O(g(x, \varepsilon)) \quad \varepsilon \to 0$$

may hold true for certain fixed $x$ values.

If they hold true for all $x \in D$ then they are said to hold uniformly on $D$.

**Example** Let $f(x, \varepsilon) = \varepsilon^2 \sin x$, $g(x, \varepsilon) = \varepsilon x$

(1) $\varepsilon^2 \sin x \ll \varepsilon x$ as $\varepsilon \to 0$

on any set $D$ not containing $x = 0$, since

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^2 \sin x}{\varepsilon x} = \lim_{\varepsilon \to 0} \varepsilon \left(\frac{\sin x}{x}\right) = 0$$

so long as $x \neq 0$.

So for instance,

$$\varepsilon^2 \sin x \ll \varepsilon x \quad \text{uniformly on } D = [\frac{1}{2}, 2]$$
EXAMPLE Consider
\[ f(x, \varepsilon) = x + e^{-x/\varepsilon} \quad g(x, \varepsilon) = x \]
where \( D = [0, 1] \). So long as \( x \neq 0 \)
\[ \lim_{\varepsilon \to 0^+} \frac{f(x, \varepsilon)}{g(x, \varepsilon)} = \lim_{\varepsilon \to 0^+} \frac{x + e^{-x/\varepsilon}}{x} = L = 1 \]
so that for \( x \neq 0 \)
\[ (1) \quad f(x, \varepsilon) = O(g(x, \varepsilon)) \quad \text{as} \quad \varepsilon \to 0^+ \]
The statement (1) is uniformly true on \([\frac{1}{2}, 1]\), say but isn’t on \([0, 1] \). To see why, we ask does
\[ f(0, \varepsilon) = O(g(0, \varepsilon)) \quad ? \]
The answer is no. If it were true there would be a constant \( M \) such that
\[ |f(0, \varepsilon)| \leq M |g(0, \varepsilon)| \]
\[ |1 + e^0| \leq M \cdot 0 \]
\[ 1 \leq 0 \]
a contradiction. Thus (1) depends on \( x \) and does not apply uniformly on \([0, 1]\).

Comment The reasons for concern about "uniformity" are deep. See "Introduction to Perturbation Methods" M.H. Holmes for a good discussion.
Asymptotic Sequences

Defn \( \{ \phi_n(\varepsilon) \} \) is an asymptotic sequence if
\[
\phi_{n+1} \ll \phi_n \quad \text{for all } n \geq 0
\]

In words an asymptotic sequence is a sequence of asymptotically ordered \( \phi_n \).

Ex
\[
\phi_n(\varepsilon) = \varepsilon^n, \quad n \geq -2
\]
\[
\frac{1}{\varepsilon^2} \gg \frac{1}{\varepsilon} \gg 1 \gg \varepsilon \gg \varepsilon^2 \gg \ldots
\]

Ex
\[
\{ \phi_n \} = \{ \log \varepsilon, 1, \varepsilon \log \varepsilon, \varepsilon, \varepsilon^2 (\log \varepsilon)^2, \varepsilon^2 \}.
\]
\[
\{ \phi_n \} = \{ \phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5 \}.
\]

In particular note \( \phi_4(\varepsilon) \ll \phi_3(\varepsilon) \), i.e.
\[
\frac{\phi_4}{\phi_3} = \frac{\varepsilon^2 (\log \varepsilon)^2}{\varepsilon} = \varepsilon (\log \varepsilon)^2 \rightarrow 0
\]

where the latter vanishes \( (\varepsilon \rightarrow 0) \) using L'Hopital's rule

\[
\lim_{\varepsilon \rightarrow 0} \varepsilon (\log \varepsilon)^2 = \lim_{\varepsilon \rightarrow 0} \frac{(\log \varepsilon)^2}{\varepsilon^{-1}} = \lim_{\varepsilon \rightarrow 0} \frac{2 \varepsilon^{-1} \log \varepsilon}{-\varepsilon^{-2}} = \lim_{\varepsilon \rightarrow 0} -2 \varepsilon \log \varepsilon = 0 \quad \text{by prev. ex (L'Hopital)}
\]
Asymptotic approximations

Let \( \{ \phi_n(x, \varepsilon) \} \) be an asymptotic sequence of "gauge functions".

The series

\[
S = \sum_{n=0}^{\infty} a_n \phi_n(x, \varepsilon) \quad a_n \in \mathbb{R}
\]

is said to be an asymptotic approximation of \( y(x, \varepsilon) \) if

\[
y(x, \varepsilon) - \sum_{n=0}^{N} a_n \phi_n(x, \varepsilon) = o(\phi_{N+1}(x, \varepsilon))
\]

for all \( N \geq 0 \).

Remarks on error

Equation (2) is a statement about relative (not absolute) error.

EXAMPLE \( \phi_0(\varepsilon) = \frac{1}{\varepsilon^2}, \phi_1(\varepsilon) = \frac{1}{\varepsilon}, y = \frac{1}{\varepsilon^2} + \frac{1}{\sqrt{\varepsilon}} \)

Note

\[
\frac{y - \phi_0}{\phi_1} = \sqrt{\varepsilon} \to 0 \quad \text{(relative error)}
\]

so that \( y - \phi_0 = o(\phi_1) \) but that

\[
|y - \phi_0| = \frac{1}{\sqrt{\varepsilon}} \to \infty \quad \text{absolute error}
\]
Notation

Once the "quage functions" \( \phi_n(x, \varepsilon) \) have been stipulated one can denote truncated asymptotic approximations and errors, i.e.,

\[
y(x, \varepsilon) \sim a_0 \phi_0(x, \varepsilon) + o(\phi_1)
\]

"asymptotic to" with an error of this order.

which precisely means the following relative error is small

\[
\frac{y(x, \varepsilon) - a_0 \phi_0(x, \varepsilon)}{\phi_1(x, \varepsilon)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0
\]

Nonuniqueness of representation

The same function \( y(x, \varepsilon) \) can be represented with different asymptotic series:

\[
\ln(1 + \varepsilon) \sim \varepsilon + o(\varepsilon^2)
\]
\[
\ln(1 + \varepsilon) \sim \sin \varepsilon + o(\varepsilon^2)
\]

Divergent Series Not discussed/proven but it's possible for

\[
y(x, \varepsilon) \sim S = \sum_{n=0}^{\infty} a_n \phi_n(x, \varepsilon)
\]
even though \( S \) is a divergent series.