

EXAMPLE Find the extrema of

$$(1) \quad J(y) = \frac{1}{12} \int_0^1 (y')^2 dx$$

over

$$A = \{y \in C^2[0, 1] : y(0) = 0, y(1) = 1\}$$

subject to the constraint

$$(2) \quad K(y) = \int_0^1 x y(x) dx = 1$$

A necessary condition for extrema is that the augmented Lagrangian

$$(3) \quad L^*(x, y, y') = \frac{1}{12} (y')^2 + \lambda x y$$

satisfy the Euler-Lagrange Equation

$$(4) \quad L_y^* = \frac{d}{dx} L_{y'}^*$$

For L^* defined in (3) this is, explicitly,

$$\bar{y}'' = 6\lambda x$$

whose general solution is

$$(5) \quad \bar{y}(x) = \lambda x^3 + c_1 x + c_2$$

Goal is to find constants c_1 , c_2 and λ so that $y(x)$ satisfies both boundary conditions and the integral constraint.

Boundary conditions

$$\bar{y}(0) = c_2 = 0$$

$$\bar{y}(1) = \lambda + c_1 + c_2 = 1$$

Thus

$$c_1 = 1 - \lambda \quad c_2 = 0$$

and

$$(6) \quad \bar{y}(x) = \lambda x^3 + (1 - \lambda)x$$

Must choose λ so that $K(\bar{y}) = 1$

$$\begin{aligned} K(\bar{y}) &= \int_0^1 x \bar{y}(x) dx \\ &= \int_0^1 \lambda x^4 + (1 - \lambda)x^2 dx \\ &= \frac{1}{5} \lambda + \frac{1}{3} (1 - \lambda) \end{aligned}$$

Require

$$K(\bar{y}) = \frac{1}{3} - \frac{2}{15} \lambda = 1$$

yields $\lambda = -5$ so that (from (6) above)

$$\bar{y}(x) = -5x^3 + 6x$$