Iso-perimetric Problems - Motivation

Dido's Problem

Of all curves $y(x)$ connecting $A$ and $B$ having a fixed length, which maximizes the area?

$$\max_{y \in A} \int_a^b y(x) dx \quad \text{s.t.} \quad \int_a^b \sqrt{1+y'^2} \, dx = L$$

Hanging Cable

The shape of a cable of length $L$ is that which minimizes its potential energy.

$$\min_{y \in A} \int_a^b pg\, y \sqrt{1+y'^2} \, dx \quad \text{s.t.} \quad \int_a^b \sqrt{1+y'^2} \, dx = L$$

where $g =$ gravity and $p =$ lineal cable density.
Isoperimetric Problems - General Theory

In a standard class of isoperimetric problems one seeks a minimum or maximum of a functional

\[ J(y) = \int_a^b L(x, y, y') \, dx \]

over a set of functions \( A \) satisfying

\[ y(a) = y_1, \quad y(b) = y_2 \]

\[ K(y) = \int_a^b G(x, y, y') \, dx = L \]

where \( L \) is some constant. Eqn (3) is the "isoperimetric" constraint.

The set of admissible functions \( A \) is

\[ A = \{ y \in C^2[a, b] : y(a) = y_1, y(b) = y_2, \int_a^b G(x, y, y') \, dx = L \} \]

More generally, boundary conditions (2) can be altered and there may be several "isoperimetric" constraints like (3).

Define

\[ \bar{A} = \{ h \in C^2[a, b] : h(a) = 0, h(b) = 0 \} \]

This is not (quite) the set of admissible variations but if \( y \in A \) then \( y(x) + \epsilon h(x) \) still satisfies the boundary conditions (2).
If \( \bar{y}(x) \in A \) were the solution of the isoperimetric problem and \( h(x) \in \overline{A} \),

\[
K(\bar{y} + \varepsilon h) = \varphi(\varepsilon) = \int_a^b G(x, \bar{y} + \varepsilon h, \bar{y}' + \varepsilon h') \, dx
\]

need not be constant for small \( \varepsilon \).

Thus \( \bar{y} + \varepsilon h \notin A \) and \( S_{\bar{y}} = \varepsilon h \) is not an admissible variation.

To circumvent this technicality we consider variations of the form

\[
Z(x) = \bar{y}(x) + \varepsilon_1 h_1(x) + \varepsilon_2 h_2(x)
\]

where \( h_k \in \overline{A} \) satisfy the homogeneous B.C.

\[
h_k(a) = 0 \quad h_k(b) = 0
\]

Here \( \varepsilon_1, \varepsilon_2 \) are (small) parameters.

Note that when \( \varepsilon_1 = \varepsilon_2 = 0 \), \( Z(x) = \bar{y}(x) \).

Now define the real valued fns \( f: \mathbb{R}^2 \to \mathbb{R} \) and \( g: \mathbb{R}^2 \to \mathbb{R} \) by

\[
f(\varepsilon_1, \varepsilon_2) \equiv J(Z) = \int_a^b L(x, Z, Z') \, dx
\]

\[
g(\varepsilon_1, \varepsilon_2) \equiv K(Z) = \int_a^b G(x, Z, Z') \, dx
\]
In general \( z(x) \not\in A \) for all \( (\varepsilon_1, \varepsilon_2) \) near \( (0,0) \) but it is for a curve thru the origin

For \( z(x) \) to be in \( A \) it must satisfy the integral constraint \( K(z) = L \), or

\[
g(\varepsilon_1, \varepsilon_2) = L
\]

Since \( g(0,0) = L \), if \( \frac{\partial}{\partial \varepsilon} (0,0) \neq 0 \) the implicit function theorem guarantees there is some function \( \phi \) such that

\[
g(\varepsilon_1, \phi(\varepsilon_1)) = L \quad \varepsilon_1 \text{ near } 0
\]

Now we pull this all together:

\( \Phi \in A \) is a solution of the isoperimetric problem

\[
(7) \quad \min_{y \in A} J(y) \quad \text{s.t.} \quad K(y) = L
\]

only if the solution of the Lagrange multiplier problem

\[
(8) \quad \min_{\varepsilon_1, \varepsilon_2} f(\varepsilon_1, \varepsilon_2) \quad \text{s.t.} \quad g(\varepsilon_1, \varepsilon_2) = L
\]

occurs at \( (\varepsilon_1, \varepsilon_2) = (0,0) \).
Thus, the must be a hagrange multiplier $\lambda$ s.t.
\[
\frac{\partial f}{\partial \varepsilon_k}(0,0) = \lambda \frac{\partial g}{\partial \varepsilon_k}(0,0) \quad k=1,2.
\]

Written another way,
\[
(9) \quad \frac{\partial}{\partial \varepsilon_k} \left( f(\varepsilon_1,\varepsilon_2) - \lambda g(\varepsilon_1,\varepsilon_2) \right) \bigg|_{(0,0)} = 0
\]

This can all be simplified by defining an augmented hagrangeian

\[
(10) \quad L^*(x,y,y') \equiv L(x,y,y') - \lambda \cdot G(x,y,y')
\]

from the integrands of the functionals $J(y)$, $K(y)$. Then
\[
(11) \quad J^*(y) \equiv \int_a^b L^*(x,y,y') \, dx
\]

The calculus problem (9) then reduces to
\[
(12) \quad \frac{\partial}{\partial \varepsilon_k} J^*(\tilde{y} + \varepsilon_1 h_1 + \varepsilon_2 h_2) \bigg|_{(0,0)} = 0
\]

These eqns and $K(\tilde{y}) = L$ are the necessary conditions for $\tilde{y}$ to solve the isoperimetric problem.
Defining
\[ F^*(\varepsilon_1, \varepsilon_2) = \int (\bar{y} + \varepsilon_1 h_1 + \varepsilon_2 h_2) \]
using the fact \( h_k \) vanish at \( x = a, b \) it is easily shown
\[ \frac{dF^*(0,0)}{d\varepsilon_k} = \int_a^b (L^*_y \frac{d}{dx} L^*_y) h_k(x) \, dx \]
must therefore vanish since \( h_k \in \mathcal{A} \) arbitrary.

yields the augmented EL eqns
\[ L^*_y = \frac{d}{dx} L^*_y' \]

Converting this all back to the original integrands

| (I) | \( (L^*_y - \frac{d}{dx} L^*_y') = \lambda (G^*_y - \frac{d}{dx} G^*_y') \) |
| (II) | \( \bar{y}(a) = y_1 \quad \bar{y}(b) = y_2 \) |
| (III) | \( K(\bar{y}) = \int_a^b G(x, \bar{y}, \bar{y}') \, dx = L \) |

Thus to find the solution one first solves the BVP (I)-(II) for \( \bar{y}(x, \lambda) \). The solution involves an unknown \( \lambda \) which is found by requiring (III) be satisfied.
EXAMPLE  Dido's Problem

\[ J(y) = \int_0^1 y(x) \, dx = \text{area} \]

\[ K(y) = \int_0^1 \sqrt{1 + y'^2} \, dx = L \]

Maximize the area \( J(y) \) over all \( y(x) \in C^2[0,1] \)
satisfying the boundary conditions

\[ y(0) = 0 \quad y(1) = 0 \]

and the perimetric constraint

\[ K(y) = \int_0^1 \sqrt{1 + y'^2} \, dx \]

Here

\[ L(x, y, y') = y \]

\[ G(x, y, y') = \sqrt{1 + y'^2} \]

Thus the Euler lagrange eqn

\[ Ly - \frac{d}{dx} L y' = \lambda (G_y - \frac{d}{dx} G y') \]

becomes

\[ 1 = -\lambda \frac{d}{dx} \left( \frac{y'}{\sqrt{1 + y'^2}} \right) \]
Integrating in $x$

\[(1) \quad \frac{y'}{\sqrt{1+y'^2}} = -\frac{1}{\lambda} x + C_1, \quad C_1 \in \mathbb{R}\]

This is not easy to solve directly.

Claim the solution is part of a circular arc

\[(x - \frac{1}{2})^2 + (y - B)^2 = r^2\]

\[
\begin{array}{c}
\text{\scriptsize{\((x, B)\)}} \\
\end{array}
\]

To satisfy the BC at $x = 0$ and $x = 1$

\[
\frac{1}{4} + (0 - B)^2 = r^2
\]

Hence (solving for $r^2$) we find (for $B > 0$)

\[y(x) = B + \sqrt{r^2 - (x - \frac{1}{2})^2} > 0\]

\[y(x) = B + \sqrt{B^2 - x^2 + x}\]

After some calculations one can substitute $y(x)$ in (2) into (1) and square both sides to get
\[
\frac{(2x-1)^2}{4B^2+1} = \left( c_1 - \frac{1}{\lambda} x \right)^2 \quad \otimes
\]

Thus \( y(x) \) in (2) satisfies the Eqn (1) and the B.C. \( y(0) = y(1) = 0 \) if \( c_1 \) and \( B \) are chosen so \( \otimes \) is satisfied for all \( x \in (0, 1) \)

\[
\frac{2x-1}{\sqrt{4B^2+1}} = \frac{1}{\lambda} x - c_1
\]

Matching coefficients of powers of \( x \)

\[
\lambda = \frac{1}{2} \sqrt{4B^2+1}
\]

\[
c_1 = (4B^2+1)^{-\frac{1}{2}}
\]

Again, Eqns are satisfied \( \forall x \in (0, 1) \) for these choice of parameters and all \( B > 0 \)

The value of \( B \) is found by requiring the integral constraint be satisfied

\[
K \int_{a}^{b} y^2 \, dx = \int \sqrt{1 + y'^2} \, dx = \psi(B) = L
\]

The latter is a nonlinear algebraic eqn:

\[
\psi(B) = \int_{0}^{1} \sqrt{\frac{4B^2+1}{4(B^2-x^2)+x}} \, dx
\]
Explicitly, letting $\Delta = \sqrt{4B^2 + 1}$

$$\psi(B) = \frac{1}{2} \Delta \int_0^1 \frac{dx}{\sqrt{B^2 - x^2 + x}}$$

$$\psi(B) = \frac{1}{2} \Delta \arctan \left( \frac{x - \frac{1}{2}}{\sqrt{B^2 - x^2 + x}} \right) \bigg|_{x=0}^{x=1}$$

$$\psi(B) = \sqrt{4B^2 + 1} \cdot \arctan \left( \frac{1}{2B} \right)$$

Graph:

Gives the $B^*$ value for the solution of the isoperimetric problem

$$\bar{y}(x) = B^* + \sqrt{B^* - x^2 + x}$$

satisfies E. E. g. n., B.C. and integral constraint.