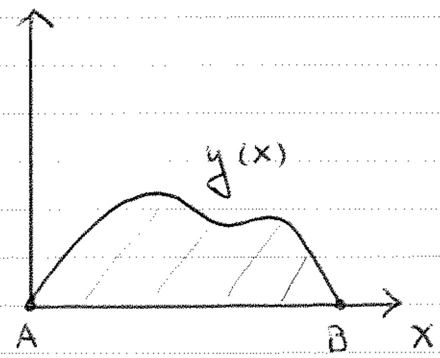
Isoperimetric Problems - Motivation

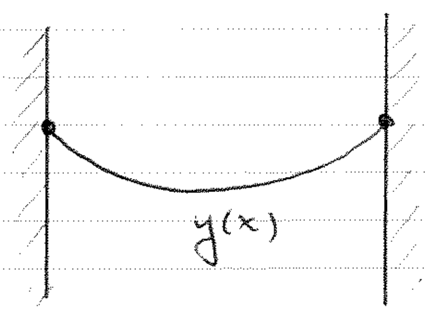
Dido's Problem



Of all curves $y(x)$ connecting A and B having a fixed length, which maximizes the area?

$$\max_{y \in A} \int_a^b y(x)\,dx \qquad \text{s.t.} \int_a^b \sqrt{1+y'^2}\,dx = L$$

Hanging Cable



The shape of a cable of length L is that which minimizes its potential energy.

$$\min_{y \in A} \int_a^b \rho g y \sqrt{1+y'^2}\,dx \qquad \text{s.t.} \int_a^b \sqrt{1+y'^2}\,dx = L$$

where g = gravity and ρ = lineal cable density

Isoperimetric Problems - General Theory

In a standard class of isoperimetric problems one seeks a minimum or maximum of a functional

$$(1) \qquad J(y) = \int_a^b L(x, y, y')\, dx$$

over a set of functions A satisfying

$$(2) \qquad y(a) = y_1 \qquad y(b) = y_2$$

$$(3) \qquad K(y) \equiv \int_a^b G(x, y, y')\, dx = L$$

where L is some constant. Eqn (3) is the "isoperimetric" constraint.

The set of admissible functions A is

$$A = \left\{ y \in C^2[a,b] : y(a) = y_1,\ y(b) = y_2,\ \int_a^b G(x, y, y')\,dx = L \right\}$$

More generally boundary conditions (2) can be altered and there may be several "isoperimetric" constraints like (3).

Define

$$\bar{A} = \left\{ h \in C^2[a,b] : h(a) = 0,\ h(b) = 0 \right\}$$

This is not (quite) the set of admissible variations but if $y \in A$ then $y(x) + \varepsilon h(x)$ still satisfies the boundary conds (2).

If $\bar{y}(x) \in A$ were the solution of the isoperimetric prblem and $h(x) \in \bar{A}$

$$K(\bar{y}+\varepsilon h) = g(\varepsilon) = \int_a^b G(x, \bar{y}+\varepsilon h, \bar{y}'+\varepsilon h')\,dx$$

need not be constant for small ε.
Thus $\bar{y}+\varepsilon h \notin A$ and $\delta y = \varepsilon h$ is not an admissible variation.

To circumvent this technicality we consider variations of the form

(4) $$z(x) = \bar{y}(x) + \varepsilon_1 h_1(x) + \varepsilon_2 h_2(x)$$

where $h_k \in \bar{A}$ satisfy the homogeneous B.C.

$$h_k(a) = 0 \qquad h_k(b) = 0$$

Here $\varepsilon_1, \varepsilon_2$ are (small) parameters.
Note that when $\varepsilon_1 = \varepsilon_2 = 0$, $z(x) = \bar{y}(x)$

Now define the real valued fns $f: \mathbb{R}^2 \to \mathbb{R}$ and $g: \mathbb{R}^2 \to \mathbb{R}$ by

(5) $$f(\varepsilon_1, \varepsilon_2) \equiv J(z) = \int_a^b L(x, z, z')\,dx$$

(6) $$g(\varepsilon_1, \varepsilon_2) \equiv K(z) = \int_a^b G(x, z, z')\,dx$$

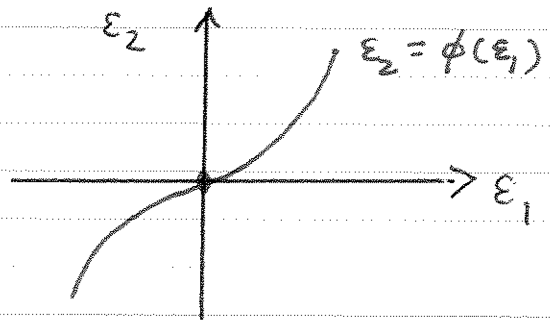
In general $z(x) \notin \mathcal{A}$ for all $(\varepsilon_1, \varepsilon_2)$ near $(0,0)$ but it is for a curve thru the origin

For $z(x)$ to be in $\mathcal{A}$ it must satisfy the integral constraint $K(z) = L$, or

$$g(\varepsilon_1, \varepsilon_2) = L$$

Since $g(0,0) = L$, if $g_{\varepsilon_k}(0,0) \neq 0$ the implicit function theorem guarantees there is some function ϕ such that

$$g(\varepsilon_1, \phi(\varepsilon_1)) = L \qquad \varepsilon_1 \text{ near } 0$$



Constraint $K(z) = L$ satisfied on this curve.

Now we pull this all together:

$\bar{y} \in \mathcal{A}$ is a solution of the isoperimetric problem

$$(7) \qquad \min_{y \in \mathcal{A}} J(y) \qquad \text{s.t.} \qquad K(y) = L$$

only if the solution of the Lagrange multiplier problem

$$(8) \qquad \min f(\varepsilon_1, \varepsilon_2) \qquad \text{s.t.} \qquad g(\varepsilon_1, \varepsilon_2) = L$$

occurs at $(\varepsilon_1, \varepsilon_2) = (0,0)$.

Thus, the must be a Lagrange multiplier λ s.t.

$$\frac{\partial f}{\partial \varepsilon_K}(0,0) = \lambda \frac{\partial g}{\partial \varepsilon_K}(0,0) \qquad K=1,2.$$

Written another way

$$(9) \qquad \frac{\partial}{\partial \varepsilon_K}\left(f(\varepsilon_1,\varepsilon_2) - \lambda g(\varepsilon_1,\varepsilon_2)\right)\Bigg|_{(0,0)} = 0$$

This can all be simplified by defining an augmented Lagrangian

$$\boxed{(10) \qquad L^*(x,y,y') \equiv L(x,y,y') - \lambda G(x,y,y')}$$

from the integrands of the functionals $J(y)$, $K(y)$. Then

$$(11) \qquad J^*(y) \equiv \int_a^b L^*(x,y,y')\,dx$$

The calculus problem (9) then reduces to

$$(12) \qquad \frac{\partial}{\partial \varepsilon_K} J^*(\bar{y}+\varepsilon_1 h_1+\varepsilon_2 h_2)\Bigg|_{(0,0)} = 0$$

These eqns and $K(\bar{y}) = L$ are the necessary conditions for $\bar{y}$ to solve the isoperimetric problem.

Defining

$$F^*(\varepsilon_1, \varepsilon_2) \equiv J^*(\bar{y} + \varepsilon_1 h_1 + \varepsilon_2 h_2)$$

using the fact h_k vanish at $x = a, b$ it is easily shown

$$\frac{\partial F^*}{\partial \varepsilon_k}(0,0) = \int_a^b \underbrace{\left(L^*_y - \frac{d}{dx} L^*_{y'}\right)}_{\text{must therefore vanish since } h_k \in \bar{A} \text{ arbitrary.}} h_k(x)\, dx$$

yields the augmented EL eqns

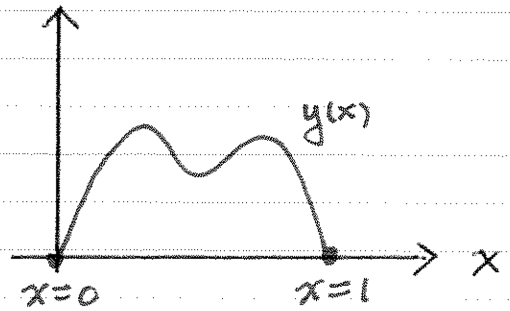
$$L^*_y = \frac{d}{dx} L^*_{y'}$$

Converting this all back to the original integrands

$$\boxed{\begin{array}{ll} (\mathrm{I}) & \left(L_y - \frac{d}{dx} L_{y'}\right) = \lambda \left(G_y - \frac{d}{dx} G_{y'}\right) \\ (\mathrm{II}) & \bar{y}(a) = y_1 \qquad \bar{y}(b) = y_2 \\ (\mathrm{III}) & K(\bar{y}) = \int_a^b G(x, \bar{y}, \bar{y}')\, dx = L \end{array}}$$

Thus to find the solution one first solves the BVP (I)-(II) ~~for~~ $\bar{y}(x, \lambda)$. The solution involves an unknown λ which is found by requiring (III) be satisfied.

EXAMPLE Dido's Problem



$$J(y) = \int_0^1 y(x)\,dx = \text{area}$$

$$K(y) = \int_0^1 \sqrt{1+y'^2}\,dx = L$$

Maximize the area $J(y)$ over all $y(x) \in C^2[0,1]$ satisfying the boundary conditions

$$y(0) = 0 \qquad y(1) = 0$$

and the perimetric constraint

$$K(y) = \int_0^1 \sqrt{1+y'^2}\,dx$$

Here

$$L(x,y,y') = y$$
$$G(x,y,y') = \sqrt{1+y'^2}$$

Thus the Euler Lagrange eqn

$$L_y - \frac{d}{dx} L_{y'} = \lambda \left(G_y - \frac{d}{dx} G_{y'} \right)$$

becomes

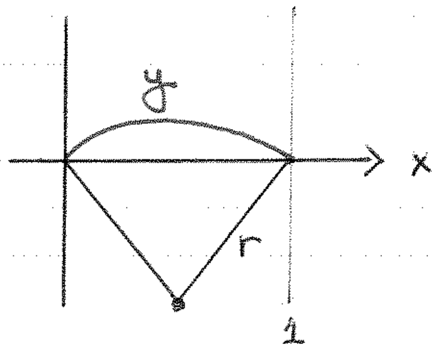
$$1 = -\lambda \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}} \right)$$

Integrating in x

$$(1) \qquad \frac{y'}{\sqrt{1+y'^2}} = -\frac{1}{\lambda}x + c_1 \qquad c_1 \in \mathbb{R}$$

This is not easy to solve directly.

Claim the solution is part of a circular arc



$$(x-\tfrac{1}{2})^2 + (y-B)^2 = r^2$$

$(\tfrac{1}{2}, B)$

To satisfy the BC at $x=0$ and $x=1$

$$\frac{1}{4} + (0-B)^2 = r^2$$

Hence (solving for r^2) we find (for $B>0$)

$$y(x) = B + \sqrt{r^2 - (x-\tfrac{1}{2})^2} > 0$$

$$(2) \qquad y(x) = B + \sqrt{B^2 - x^2 + x}$$

After some calculations one can substitute $y(x)$ in (2) into (1) and square both sides to get

$$\boxed{\frac{(2x-1)^2}{4B^2+1} = \left(c_1 - \frac{1}{\lambda}x\right)^2} \quad (*)$$

Thus $y(x)$ in (2) satisfies the EL eqn (1) and the B.C. $y(0) = y(1) = 0$ if c_1 and B are chosen so $(*)$ is satisfied for all $x \in (0,1)$

$$\frac{2x-1}{\sqrt{4B^2+1}} = \frac{1}{\lambda}x - c_1$$

Matching coefficients of powers of x

$$\lambda = \tfrac{1}{2}\sqrt{4B^2+1}$$

$$c_1 = (4B^2+1)^{-1/2}$$

Again, EL eqns are satisfied $\forall x \in (0,1)$ for these choice of parameters and all $B > 0$ for

$$\bar{y}(x) = B + \sqrt{B^2 - x^2 + x}$$

The value of B is found by requiring the integral constraint be satisfied

$$K(\bar{y}) = \int_0^1 \sqrt{1+\bar{y}'^2}\,dx = \psi(B) = L$$

The latter is a nonlinear algebraic eqn:

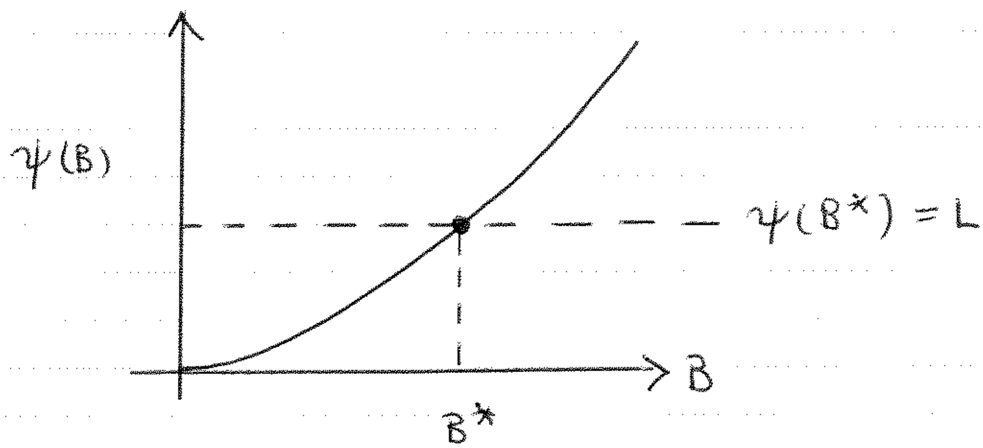
$$\psi(B) = \int_0^1 \sqrt{\frac{4B^2+1}{4(B^2-x^2+x)}}\,dx$$

Explicitly, letting $\Delta = \sqrt{4B^2+1}$

$$\psi(B) = \tfrac{1}{2}\Delta \int_0^1 \frac{dx}{\sqrt{B^2 - x^2 + x}}$$

$$\psi(B) = \tfrac{1}{2}\Delta \; \arctan\left(\frac{x - \frac{1}{2}}{\sqrt{B^2 - x^2 + x}} \right) \Bigg|_{x=0}^{x=1}$$

$$\boxed{\psi(B) = \sqrt{4B^2+1} \;\; \arctan\left(\frac{1}{2B} \right)}$$



Gives the B^* value for the soln of the isoperimetric problem

$$\bar{y}(x) = B^* + \sqrt{B^{*} - x^2 + x}$$

satisfies EL eqn, B.C. and integral constraint.