

Principle of Least Action

Let $q \in \mathbb{R}^n$ be a coordinate system for some physical system and:

$T(q, \dot{q}) =$ kinetic energy

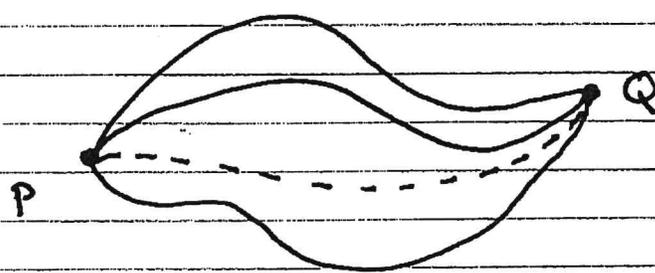
$V(q, \dot{q}) =$ potential energy

More generally V may depend on time t .

$$L \equiv T - V \quad \text{Lagrangian}$$

Then the path followed by a (holonomic) system on $[t_1, t_2]$ is that which extremizes the action

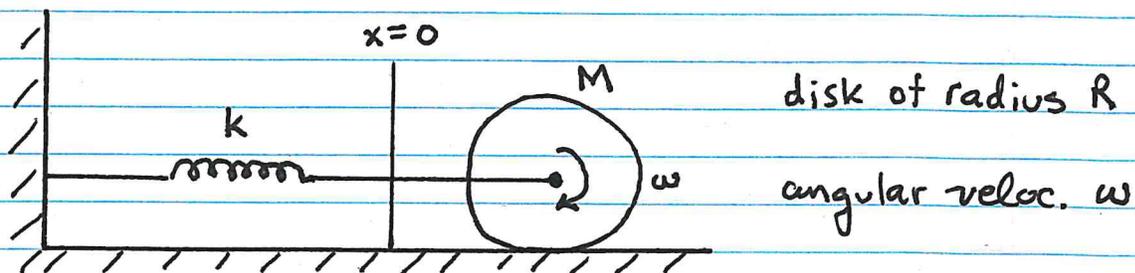
$$J(q) \equiv \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$$



----- extrema

Fixed endpoints

Rolling Disk - Least Action



$$I = \frac{1}{2} MR^2 \quad \text{moment of inertia}$$

$$\omega = \frac{v}{R} \quad \text{angular velocity}$$

Derive Lagrangian

$$L = \text{K.E.} - \text{P.E.}$$

$$L = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 - \frac{1}{2} k x^2$$

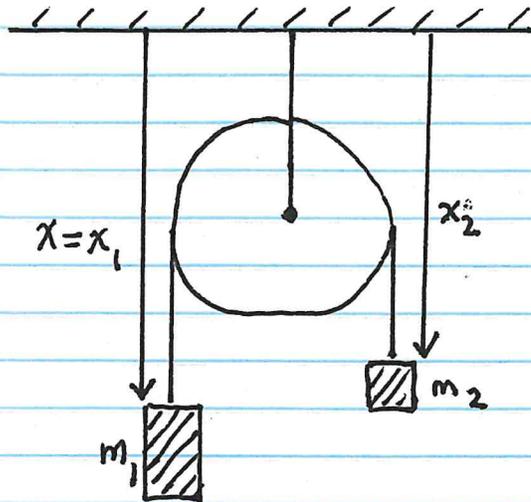
$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \left(\frac{1}{2} MR^2 \right) \left(\frac{v^2}{R^2} \right) - \frac{1}{2} k x^2$$

$$L = \frac{3}{4} M \dot{x}^2 - \frac{1}{2} k x^2$$

Euler-Lagrange eqns

$$\ddot{x} = - \left(\frac{8k}{3M} \right) x$$

Atwood Machine - Least Action



l = length of cable connecting m_1 and m_2

$x = x_1$
 $l - x = x_2$) holonomic constraint

Derive Lagrangian

$$L = \text{K.E.} - \text{P.E.}$$

$$L = \underbrace{\frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2}_{\text{K.E.}} + \underbrace{m_1 g x_1 + m_2 g x_2}_{-\text{P.E.}}$$

$$L = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + m_1 g x + m_2 g (l - x)$$

Euler-Lagrange Eqns

$$\ddot{x} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) g$$

Conjugate Momenta and Generalized Force

For $q = (q_1, \dots, q_n)$ the EL-eqns which extremize the action are

$$(1) \quad \frac{\partial L}{\partial q_i} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right)$$

For each coordinate q_i we define a conjugate momenta p_i and generalized force F_i by

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

$$F_i \equiv \frac{\partial L}{\partial q_i}$$

Then the equations of motion (1) become

$$\frac{d\vec{p}}{dt} = \vec{F}$$

which is very similar to Newton's Law.

Common Kinetic Energy Coordinates

EX Cartesian $q = (x, y, z)$

$$T = \frac{1}{2} m \dot{q}^T \dot{q} = \frac{1}{2} m \dot{q}_i^2 \quad (\text{sum})$$

EX Cylindrical $q = (r, \theta, z)$

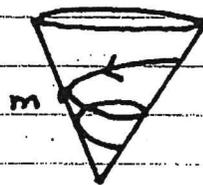
$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2)$$

EX Spherical $q = (\rho, \phi, \theta)$

$$T = \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \sin^2 \phi \dot{\theta}^2)$$

Lagrangian Mechanics

Often systems have constraints (holonomic) which functionally related coordinates



$$\phi(q) = z^2 - x^2 - y^2 = 0$$

on cone

So a more general class of problems in Lagrangian Mechanics is

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}$$

$$i = 1, \dots, n$$

$$\phi_j(q, t) = 0$$

$$j = 1, \dots, m \leq n$$

Later are "holonomic" constraints.

EXAMPLE Planar motion with central potential

Let (r, θ) be polar coordinates and define

$$L(q, \dot{q}) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$$

Since $L_{\theta} = 0$ the momenta $p_{\theta} \equiv L_{\dot{\theta}} = mr^2 \dot{\theta}$ conjugate to θ is conserved:

$$\frac{dp_{\theta}}{dt} = 0 \quad p_{\theta} = mr^2 \dot{\theta}$$

This is conservation of angular momentum!

The second EL-eqn $L_r = \frac{d}{dt} L_{\dot{r}}$ implies (then)

$$(1) \quad m \ddot{r} = \frac{p_{\theta}^2}{mr^3} - V'(r)$$

Solving the equations of motion reduces to the integrability of (1) of the general form

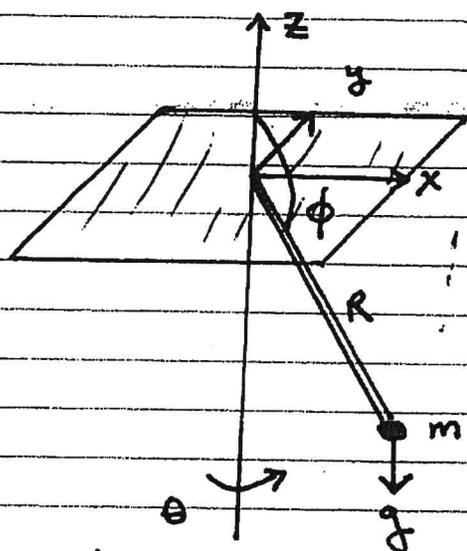
$$\ddot{r} + G(r) = 0$$

Given (1), one can then prove energy E is also conserved

$$\frac{dE}{dt} = \frac{d}{dt} (T + V) = \left(m\ddot{r} - \frac{p_{\theta}^2}{mr^3} + V'(r) \right) \dot{r} = 0$$

EXAMPLE

Spherical Pendulum (length R)



Holonomic constraint

$$x^2 + y^2 + z^2 = R^2$$

In spherical (ρ, ϕ, θ) is

$$\rho = R$$

Transformations:

$$\begin{aligned}x &= R \cos \theta \sin \phi \\y &= R \sin \theta \sin \phi \\z &= R \cos \phi\end{aligned}$$

Kinetic energy $T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ using $\dot{R} = \dot{\rho} = 0$

$$T = \frac{1}{2} m (R^2 \dot{\theta}^2 \sin^2 \phi + R^2 \dot{\phi}^2)$$

Potential energy (gravity) is $U = mgz$ or

$$U = mgR \cos \phi$$

Generalized coordinates $q = (q_1, q_2) = (\theta, \phi)$

Since $L \equiv T - U$ and $L_{\theta} = 0$ its associated conjugate momentum p_{θ} is constant

(1)

$$p_{\theta} = L_{\dot{\theta}} = mR^2 \dot{\theta} \sin^2 \phi$$

Conjugate momentum p_{ϕ} is not constant,

Given

$$L_{\dot{\phi}} = P_{\dot{\phi}} = mR^2 \dot{\phi}$$

$$(2) \quad L_{\phi} = mR^2 \dot{\phi}^2 \sin \phi \cos \phi - mgR \sin \phi$$

The EL-equations are (for ϕ)

$$(3) \quad mR^2 \ddot{\phi} = L_{\phi}$$

Conservation law (1) can be used to eliminate $\dot{\phi}$ indicated in (2). Ultimately (3) reduces to

$$(4) \quad \ddot{\phi} + G(\phi) = 0$$

where

$$-G(\phi) = \left(\frac{P_0^2}{m^2 R^4} \right) \frac{\cos \phi}{\sin^3 \phi} + \frac{g}{R} \sin \phi$$

The system (4) is Hamiltonian (Dyn Sys Theory) and is principal integrable:

$$\dot{\phi} \dot{\phi} + G(\phi) \phi = 0$$

$$(5) \quad \frac{1}{2} \dot{\phi}^2 + V(\phi) = c, \quad c \in \mathbb{R}$$

for the (new) potential

$$V(\phi) = \int G(s) ds.$$

Solving (5) for $\dot{\phi}$ yields a separable non-linear first order problem!!