Necessary Conditions for Local Min (Max)

If $\bar{y}$ is a local min of J then

(i) $$\Delta J = J(\bar{y}+\varepsilon h) - J(\bar{y}) \geq 0$$

for all admissible variations $\delta y = \varepsilon h(x)$

In this setting $y = \bar{y} + \varepsilon h$ and $\bar{y}$ are near each other if ε is small since

$$\|y - \bar{y}\| = \|\varepsilon h\| = \varepsilon \|h\| \to 0 \quad \text{as } \varepsilon \to 0.$$

Since

$$\Delta J = \delta J(\bar{y}, h)\,\varepsilon + o(\varepsilon)$$

a necessary condition for $\Delta J \geq 0$ is that δJ vanish (noting though ε is small it can be positive or negative).

This argument is true for every $h(x)$ for which $\varepsilon h(x)$ is an admissible variation

Theorem $\bar{y}$ is a local min of $J : A \to \mathbb{R}$ only if

$$\delta J(\bar{y}, h) = 0$$

for all admissible variations $\delta y = \varepsilon h$.

The theorem is true for local max $\bar{y}$ as well.

<u>Theorem</u> Let $g(x) \in C[a,b]$ and suppose

$$(1) \qquad \int_a^b g(x)\, v(x)\, dx = 0 \qquad \forall v \in A^*$$

where A^* is any one of the following spaces

$$A^* = C[a,b]$$

$$A^* = C^n[a,b]$$

$$A^* = \{y : y \in C^n[a,b],\ y(a) = y(b) = 0\}$$

Then $g(x) \equiv 0$.

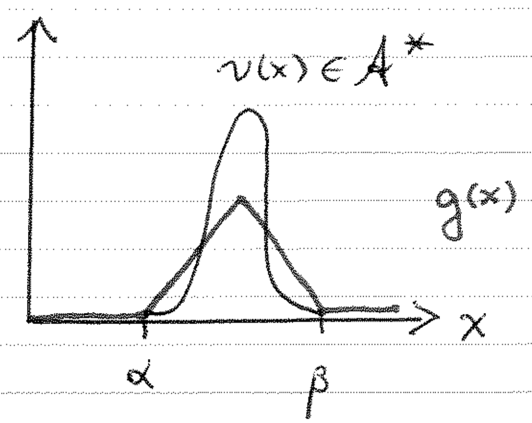
<u>Proof Outline</u> (By contradiction)

Suppose (1) is true but $g(x)$ is not identically zero. Since $g(x)$ is continuous there must be some interval $[\alpha, \beta] \subset [0, 1]$ on which it is nonzero. Without loss of generality $g(x) > 0$ on $I = [\alpha, \beta]$.

Pick any $v \in A^*$ positive on I but zero elsewhere. Then

$$\int_a^b g(x)\, v(x)\, dx = \int_\alpha^\beta g(x)\, v(x)\, dx > 0$$

contradicts (1)



Euler Lagrange Equations

Let $J : \mathcal{A} \to \mathbb{R}$ where

$$J(y) \equiv \int_a^b L(x, y(x), y'(x))\, dx$$

$$\mathcal{A} = \{ y : y \in C^2[a,b] \,,\, y(a) = y_1 ,\, y(b) = y_2 \}$$

and $L(x, y, y')$ is continuously differentiable in (x, y, y').

Then every extrema $\bar{y} \in \mathcal{A}$ of $J(y)$ satisfy the Euler Lagrange equation

$$(1) \qquad L_y(x, \bar{y}, \bar{y}') = \frac{d}{dx} L_{y'}(x, \bar{y}, \bar{y}')$$

Here 'extrema' are all potential minimizers or maximizers of $J(y)$ over $\mathcal{A}$.

The Euler-Lagrange equation (1) taken collectively with the boundary conditions form a nonlinear boundary value problem for extrema

__Proof__ Let $\bar{y} \in \mathcal{A}$ be an extrema of J and

$$F(\varepsilon) \equiv J(\bar{y} + \varepsilon h) = \int_a^b L(x, \bar{y} + \varepsilon h, \bar{y}' + \varepsilon h')\, dx$$

We seek to compute $F'(0) = \delta J(\bar{y}, h)$

$$F'(\varepsilon) = \int_a^b \frac{\partial}{\partial \varepsilon} L(x, \bar{y}+\varepsilon h, \bar{y}'+\varepsilon h')\, dx$$

$$F'(\varepsilon) = \int_a^b L_y(x,y,y')\frac{\partial}{\partial \varepsilon}(\bar{y}+\varepsilon h) + L_{y'}(x,y,y')\frac{\partial}{\partial \varepsilon}(\bar{y}'+\varepsilon h')\; dx$$

where $y \equiv \bar{y}+\varepsilon h$. Here L_y is the partial of L in y whereas $L_{y'}$ is the partial of L w.r.t. y'.

Evaluating at $\varepsilon = 0$

$$F'(0) = \int_a^b L_y(x,\bar{y},\bar{y}')h + L_{y'}(x,\bar{y},\bar{y}')h' \; dx$$

Integrate the second term by parts

$$F'(0) = \underbrace{L_{y'}(x,\bar{y},\bar{y}')h\Big|_a^b}_{\substack{\text{vanishes since } h(a)=h(b)=0 \\ \text{for } \varepsilon h(x) \text{ to be admissible variation}}} + \int_a^b \left(L_y - \frac{d}{dx}L_{y'}\right) h \, dx$$

Conclude

$$F'(0) = \delta J(\bar{y},h) = \int_a^b \Big[\underbrace{L_y(x,\bar{y},\bar{y}') - \frac{d}{dx}L_{y'}(x,\bar{y},\bar{y}')}_{g(x)}\Big]h(x)\, dx$$

For $\delta J(\bar{y},h) = 0$ for all addmissible $h \in C^2[a,b]$ the term $g(x) \in C[a,b]$ must vanish. This is the Euler Lagrange Equation!

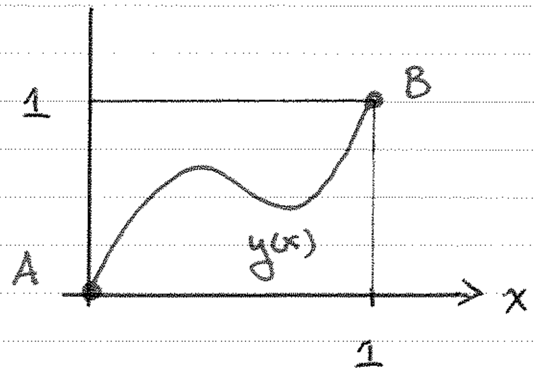
$\square$

EXAMPLE Arclength Functional

$$J(y) = \int_0^1 \sqrt{1 + y'(x)^2}\, dx$$

$$A = \{ y : y \in C^2[0,1],\ y(0)=0,\ y(1)=1 \}$$

Seek an extrema $\bar{y} \in A$ that minimizes J



Which $\bar{y}$ minimizes the distance from A to B

Here the Lagrangian $L(x, y, y') = (1 + y'^2)^{1/2}$

Euler-Lagrange equations

$$L_y(x, \bar{y}, \bar{y}') = \frac{d}{dx} L_{y'}(x, \bar{y}, \bar{y}')$$

$$(1) \qquad 0 = \frac{d}{dx}\left(\bar{y}'(1 + \bar{y}'^2)^{-1/2} \right)$$

From which we conclude there is some $C \in \mathbb{R}$ s.t.

$$(2) \qquad \bar{y}'(1 + \bar{y}'^2)^{-1/2} = C$$

One could solve eqn (2) for $\bar{y}'$ but one need only note that $\bar{y}'$ is a root of the equation

$$G(\bar{y}') = c$$

where $G(z) = z(1+z^2)^{-1/2}$. In other words,

$$(3) \qquad \bar{y}' = A$$

for some constant A. Thus

$$\bar{y}(x) = Ax + B$$

is the general soln of the E-L eqn.

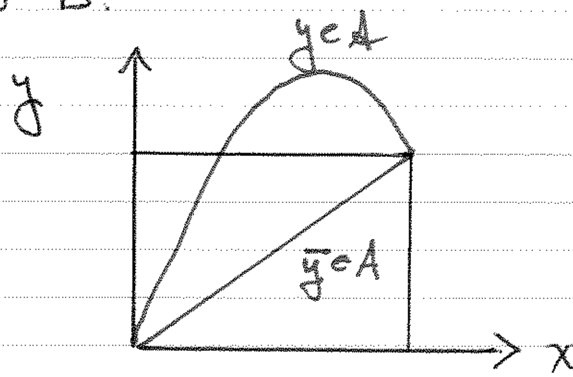
For $\bar{y} \in \mathcal{A}$ it must also satisfy B.C.

$$\left.\begin{aligned} \bar{y}(0) &= \quad\;\; B = 0 \\ \bar{y}(1) &= A + B = 1 \end{aligned}\right\} A = 1,\ B = 0$$

Hence the sole extrema is

$$\bar{y}(x) = x$$

and the minimizer is the straight line from A to B.



$J(y) \geq J(\bar{y})$

First Integrals of Euler Lagrange Eqns

Let

$$J(y) = \int_a^b L(x, y(x), y'(x))\, dx$$

For sufficiently smooth Lagrangians $L(x, y, z)$ a necessary condition that $\bar{y} \in \mathcal{A}$ extremizes J is that it satisfy the Euler Lagrange equation

$$(1) \qquad L_y(x, \bar{y}, \bar{y}') = \frac{d}{dx} L_{y'}(x, \bar{y}, \bar{y}')$$

In general (1) is a second order nonlinear differential equation and can be very difficult to solve.

However, certain special cases for solving the Euler Lagrange Equation exist

CASE ONE $\quad L = L(x, y)$ doesn't depend on y'

Then the EL-eqn simplifies to

$$(2) \qquad L(x, \bar{y}) = 0$$

which is algebraic for extrema $\bar{y}(x)$

EXAMPLE $\quad L(x, y) = e^y - x^2 y, \; \mathcal{A} = C[1, 2]$

$$L_y(x, \bar{y}) = e^{\bar{y}} - x^2 = 0$$

$$\bar{y}(x) = 2 \ln x$$

CASE TWO $L = L(x, y')$ doesn't depend on y

Then the EL-eqns simplify to

$$0 = \frac{d}{dx} L_{y'}(x, \bar{y}')$$

which implies

$$(3) \qquad L_{y'}(x, \bar{y}') = c$$

for come constant c. This is a first order differential equation for $\bar{y}(x)$

EXAMPLE

$$L(x, y') = \frac{1}{3x^2}(y')^3 - x^2 y'$$

Thus $L_{y'}(x, \bar{y}')$ is a first integral and $\bar{y}$ solves

$$L_{y'}(x, \bar{y}') = \frac{1}{x^2}(\bar{y}')^2 - x^2 = C_1$$

for some $c \in \mathbb{R}$. Solve for $\bar{y}'$

$$\bar{y}'(x) = \pm x \sqrt{C_1 + x^2}$$

Integrate to find general soln

$$\bar{y}(x) = \pm \tfrac{1}{3}(C_1 + x^2)^{3/2} + C_2$$

where C_2 is some other constant. If $\bar{y} \in \mathcal{A}$

$$\mathcal{A} = \{ y : y \in C^2[1,2], \ \bar{y}(1) = 2, \ \bar{y}(2) = 1$$

Then c_1, c_2 must solve 2 Bound. Cond.

$$\bar{y}(1) = \pm \tfrac{1}{3}(c_1 + 1)^{3/2} + c_2 = 2 \quad , \quad \bar{y}(2) = \pm \tfrac{1}{3}(c_1 + 2)^{3/2} + c_2 = 1$$

CASE THREE $L = L(x, y, y')$ doesn't depend on x

At first the EL-eqns appear intractible

$$(4) \qquad L_y(y, y') = \frac{d}{dx} L_{y'}(y, y')$$

However, one can find a first integral for (4)

Theorem If y is a solution of $$L_y(y,y') = \frac{d}{dx} L_{y'}(y,y')$$ Then $$(5) \qquad L(y,y') - y' L_{y'}(y,y') = C$$ for some constant C.

Thus, instead of solving the second order diff. eqn (4) one solves the first order eqn (5)

Pf/

$$\frac{d}{dx}(L - y' L_{y'}) = \underbrace{\frac{dL}{dx}} - y' \frac{d}{dx} L_{y'} - y'' L_{y'}$$

$$= \underbrace{L_y y' + \cancel{L_{y'} y''}} - y' \frac{d}{dx} L_{y'} - \cancel{y'' L_{y'}}$$

$$= y' \underbrace{\left(L_y - \frac{d}{dx} L_{y'} \right)}_{= 0 \text{ since } y \text{ solves EL-eqn}}$$

$$= 0$$

Thus $L - y' L_{y'}$ must be a constant □

EXAMPLE Find extrema for

$$J(y) = \int_0^{\pi/2} (y^2 - y'^2)\, dx$$

$$\mathcal{A} = \{ y : y \in C^2[0, \tfrac{\pi}{2}],\ y(0) = 0,\ y(\tfrac{\pi}{2}) = 1 \}$$

Since the Lagrian doesn't depend on x there is some constant c such that

$$(1) \qquad L - y' L_{y'} = c$$

For $L = y^2 - y'^2$ we have for extrema $\bar{y} \in \mathcal{A}$

$$(\bar{y}^2 - \bar{y}'^2) - \bar{y}'(-2\bar{y}') = c$$

$$\bar{y}'^2 + \bar{y}^2 = c_1^2 \qquad > 0$$

which is separable

$$\frac{d\bar{y}}{dx} = \pm\sqrt{c_1^2 - \bar{y}^2}$$

$$\frac{d\bar{y}}{\sqrt{c_1^2 - \bar{y}^2}} = \pm\, dx$$

Integrate and invert the $\frac{1}{c_1} \arcsin(\bar{y}/c_1)$ term:

$$\bar{y}(x) = A \sin(B + x)$$

for constants A, B that depend on c_1, c_2

Since $\bar{y} \in A$ it satisfies the Bound. Cond.

$$\bar{y}(0) = A \sin B = 0$$

$$\bar{y}(\tfrac{\pi}{2}) = A \sin(B + \tfrac{\pi}{2}) = 1$$

Cant have $A = 0$ else $\bar{y}(x) \equiv 0 \notin A$. Thus $B = 0$ and $A = 1$

$$\bar{y}(x) = \sin x$$

is the sole extrema.

<u>Remarks</u> Though the first integral

$$L - y' L_{y'} = c$$

leads to a solution, solving the E-Lag Eqns directly lead to the soln more quickly:

Since $L_y = 2y$ and $L_{y'} = -2y'$ the E-L eqns are:

$$L_y = \frac{d}{dx} L_{y'}$$

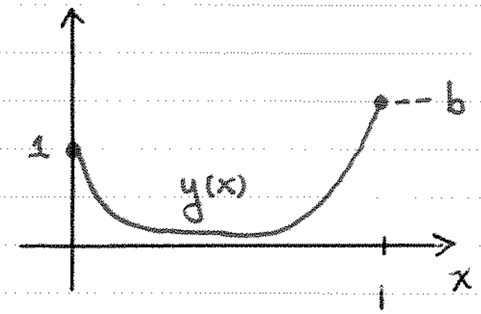
$$2y = \frac{d}{dx}(-2y') = -2y''$$

Hence

$$\bar{y}'' + \bar{y} = 0 \qquad \bar{y}(0) = 0 \qquad \bar{y}(1) = \tfrac{\pi}{2}$$

whose soln is clearly $\bar{y} = \sin x$.

<u>EXAMPLE</u> <u>Surface area of revolution</u>



Revolve the graph $y(x)$ about x-axis to form a surface

Given $y(0) = 1$, $y(1) =$ which $y(x)$ minimizes surface area?

Surprisingly, the answer is not a cone!

$$J(y) = 2\pi \int_0^1 y\sqrt{1+y'^2}\, dx$$

is surface area of revolution from calculus.

The admissible set is

$$\mathcal{A} = \{ y : y \in C^2[0,1] \,,\, y(0) = 1 \,,\, y(1) = b$$

Since the Lagrangian

$$L = L(y, y') = y\sqrt{1+y'^2}$$

does not depend on x, a first integral of the Euler Lagrange eqns is

$$L - y' L_{y'} = c$$

or,

$$y\sqrt{1+y'^2} - \frac{y y'^2}{\sqrt{1+y'^2}} = c$$

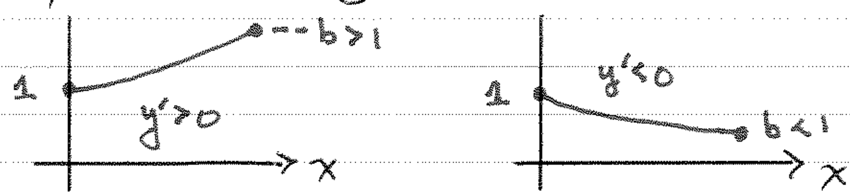
This can be algebraically be simplified to

$$\frac{y}{\sqrt{1+y'^2}} = c$$

Solving for y' we find

$$\frac{dy}{dx} = \pm\sqrt{\left(\frac{y}{c}\right)^2 - 1}$$

If $b > 1$ (without loss of generality) we may assume $y'(x) > 0$ hence "+" sign.



Then

$$\frac{dy}{dx} = +\sqrt{\left(\frac{y}{c}\right)^2 - 1}$$

which is separable. The general soln is

$$y(x) = c\cosh\left(\frac{x}{c} + \mu\right)$$

where $c, \mu \in \mathbb{R}$ are constants.

The boundary condition $y(0) = 1$ implies

$$c = (\cosh\mu)^{-1}$$

So that

$$(1) \quad y(x) = \frac{\cosh(x\cosh\mu + \mu)}{\cosh\mu}$$

Though the constant μ cannot be found explicitly, it can be found implicitly:

$$y(1) = \frac{\cosh(\cosh\mu + \mu)}{\cosh\mu} = b$$

Thus μ is the root of

$$(2) \qquad \cosh(\cosh\bar{\mu} + \bar{\mu}) = b \cosh\bar{\mu}$$

One can find, for a given b, the root of (2) numerically and use it in

$$\bar{y}(x) = \frac{\cosh(x\cosh\bar{\mu} + \bar{\mu})}{\cosh\bar{\mu}}$$

as the extrema (minimizer) for the problem.