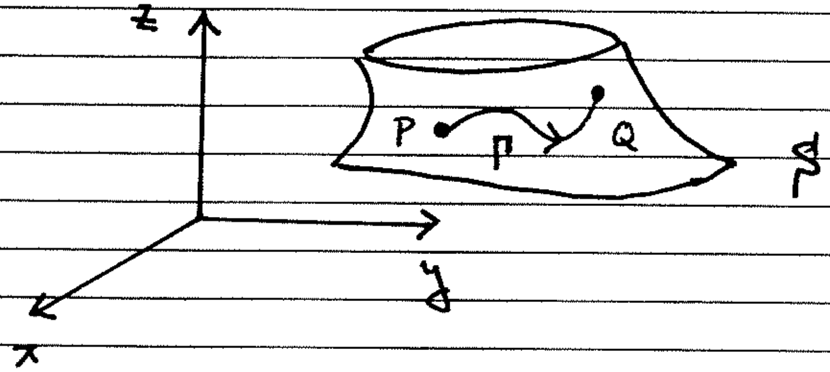
Geodesics

Consider a surface S in $\mathbb{R}^3$ with two points P and Q identified on it

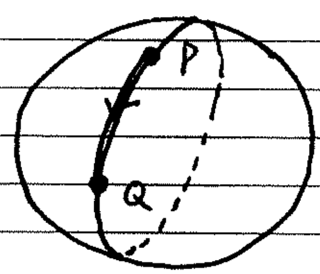


Γ = any path connecting points P and Q

Of all paths Γ that connect P and Q, the one having the shortest length is called the <u>geodesic</u> from P to Q.

There are many geodesics on surface S

The geodesics on a sphere are great circles though not easily proven (with variational calculus)



General Formulation

Surfaces S can be defined in many ways. For example, S may be the set of (x, y, z) that satisfy

$$(1) \qquad g(x, y, z) = 0$$

for some function g. If we let a curve Γ on S be defined parametrically

$$\Gamma(t) = (X(t), Y(t), Z(t)) \qquad t \in [a, b]$$

then the arc length is

$$(2) \qquad J(X, Y, Z) = \int_a^b \sqrt{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2}\, dt$$

However eqn (1) relates (X, Y, Z) and hence its derivatives

$$\frac{\partial g}{\partial x} \dot{X} + \frac{\partial g}{\partial y} \dot{Y} + \frac{\partial g}{\partial z} \dot{Z} = 0$$

can in principle be used to rewrite J in terms of X, Y alone.

Geodesics on a sphere

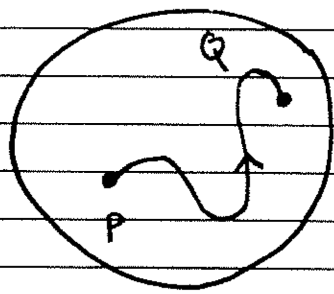
Without loss of generality a curve Γ on a sphere of radius R has the parametrization

(1) $\quad X(t) = R \sin\phi(t) \cos\theta(t)$

(2) $\quad Y(t) = R \sin\phi(t) \sin\theta(t)$

(3) $\quad Z(t) = R \cos\phi(t)$

where $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi)$ are the latitudinal and longitudinal angles respectively for $t \in [0, 1]$ $\Diamond$



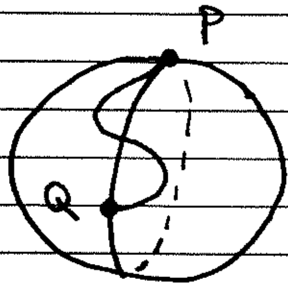
$P = (X(0), Y(0), Z(0))$

$Q = (X(1), Y(1), Z(1))$

These points are therefore defined by the initial and final angles $\theta(0), \phi(0), \theta(1), \phi(1)$.

However, to simplify matters, we may assume P is at the north pole and

$\theta(1) = 0$
$\phi(1) = \phi_1$



$P = (0, 0, R)$

$Q = (R\sin\phi_1, 0, R\cos\phi_1)$

Calculations using (1)-(3) imply

$$\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 = R^2\left(\dot{\theta}^2 \sin^2\phi + \dot{\phi}^2\right)$$

Thus the min-path problem involves finding $(\phi(t), \theta(t))$ which minimize

$$(4) \qquad J(\phi, \theta) \equiv \int_0^1 L(\phi, \dot{\phi}, \theta, \dot{\theta})\, dt$$

where

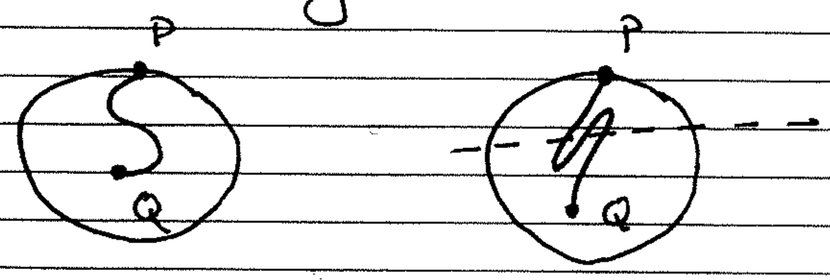
$$(5) \qquad L = R\sqrt{\dot{\theta}^2 \sin^2\phi + \dot{\phi}^2} \qquad \text{Lagrangian.}$$

One could try to solve the EL-eqns

$$(6) \qquad L_\phi = \frac{d}{dt} L_{\dot{\phi}}$$

$$(7) \qquad L_\theta = \frac{d}{dt} L_{\dot{\theta}}$$

However we may restrict our attention to those curves for which $\theta = y(\phi)$ for some function y.



$\theta = y(\phi)$ $\qquad\qquad$ $\theta \neq y(\phi)$

For this restriction (and $\dot{\phi} > 0$)

$$\dot{\theta} = y'(\phi)\,\dot{\phi}$$

so that

$$L = \dot{\phi}\sqrt{1 + \sin^2\phi\; y'(\phi)^2}$$

Then the functional J in eqn (4) is

$$J = \int_0^1 \dot{\phi}\sqrt{1 + \sin^2\phi\; y'(\phi)^2}\, dt$$

For the substitution $x = \phi(t)$, $\dot{x} = \dot{\phi}(t)$ it suffices we consider

$$(8) \quad \bar{J}(y) = \int_0^{\phi_1} \underbrace{\sqrt{1 + \sin^2\phi\; y'(\phi)^2}}_{\bar{L}(y, y', \phi)}\, d\phi$$

Since $\bar{L}_y = 0$ the EL-eqns for $\bar{J}(y)$ imply that $\bar{L}_{y'}$ is constant

$$(9) \quad \bar{L}_{y'} = \frac{(\sin\phi)^2\, y'}{\sqrt{1 + (\sin\phi)^2 y'^2}} = c$$

where $y(\phi_1) = 0$.

Claim

$$\bar{y}(\phi) \equiv 0$$

is the solution. Certainly satisfies $\bar{y}(\phi) = 0$. Also satisfies necessary condition (9) if $c = 0$.

But $\bar{y}(\phi) = 0$ for all ϕ is the great circle soln!