Lagrangians with higher order derivatives

(1) $$J(y) = \int_a^b L(x, y, y', y'')\,dx$$

where

$$\mathcal{A} = \{ y \in C^4[a,b] : y(a) = A_1, y'(a) = A_2, y(b) = B_1, y'(b) = B_2 \}$$

Admissible variations $\delta y = \varepsilon h(x)$ have

(2) $$h(a) = h'(a) = h(b) = h'(b) = 0$$

Using this assume $\bar{y} \in \mathcal{A}$ extremizes J and

$$F(\varepsilon) \equiv J(\bar{y} + \varepsilon h)$$

The first variation must vanish, i.e.

$$\delta J(\bar{y}, h) = F'(0) = \frac{d}{d\varepsilon} J(\bar{y} + \varepsilon h)\Big|_{\varepsilon = 0}$$

must vanish for all $h \in \mathcal{A}^*$ satisfying (2).

Easy to show

$$\delta J(\bar{y}, h) = \int_a^b (L_y h + L_{y'} h' + L_{y''} h'')\,dx$$

where $L_y, L_{y'}$ and $L_{y''}$ are evaluated at $(x, \bar{y}, \bar{y}', \bar{y}'')$.

Integrating by parts (twice)

$$\delta J(\bar{y}, h) = \left[L_{y''} h' - \frac{d}{dx} L_{y''} h + h L_{y'} \right] \Bigg|_{x=a}^{x=b}$$

$$+ \int_a^b \left(L_y - \frac{d}{dx} L_{y'} + \frac{d^2}{dx^2} L_{y''} \right) h(x)\, dx.$$

The first terms vanish since $h, h' = 0$ at $x = a, b$ leaving.

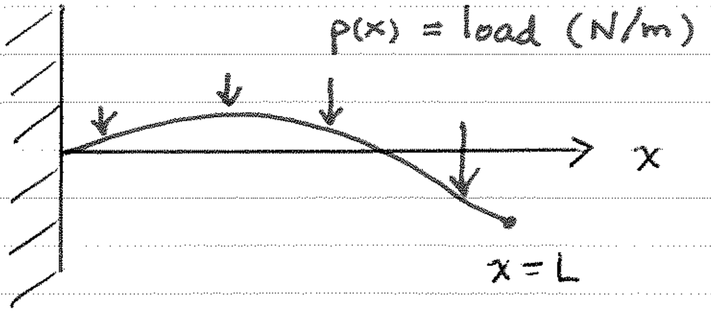
$$\delta J(\bar{y}, h) = \int_a^b \left(L_y - \frac{d}{dx} L_{y'} + \frac{d^2}{dx^2} L_{y''} \right) h(x)\, dx$$

This must vanish for all $h(x) \in A^*$ from which we conclude

$$\boxed{L_y - \frac{d}{dx} L_{y'} + \frac{d^2}{dx^2} L_{y''} = 0}$$

are the Euler Lagrange equations for this functional.

EXAMPLE Pinned Beam



Potential Energy (linear elasticity theory)

$$\Pi(y) = \int_0^L \left(\tfrac{1}{2}\mu (y'')^2 - p(x)\, y \right) dx$$

where μ = flexural rigidity.

$$L = \tfrac{1}{2}\mu (y'')^2 - p(x)\, y$$

$$L_y = -p(x)$$

$$L_{y'} = 0$$

$$L_{y''} = \mu y''$$

The Euler Lagrange Eqns

$$L_y \cancel{- \frac{d}{dx} L_{y'}} + \frac{d^2}{dx^2} L_{y''} = 0$$

become

$$y^{(4)}(x) = \frac{1}{\mu} p(x)$$

Higher Dimensional Functionals

Some functionals depend on more than one function. For instance,

$$J(y_1, y_2) = \int_a^b L(x, y_1, y_2, y_1', y_2')\, dx$$

where $y_1 \in \mathcal{A}_1$, $y_2 \in \mathcal{A}_2$ and for $k = 1, 2$

$$\mathcal{A}_k = \{ y_k \in C^4[a,b] : y_k(a) = A_k,\ y_k(b) = B_k \}$$

Here J assigns a real to each pair of functions $(y_1, y_2) \in \mathcal{A}_1 \times \mathcal{A}_2$.

$$J : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathbb{R}$$

The set $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is called the product space of $\mathcal{A}_1$ and $\mathcal{A}_2$.

To find extrema (local min/max) of J we need to consider

$$F(\varepsilon) = J(\bar{y}_1 + \varepsilon h_1, \bar{y}_2 + \varepsilon h_2)$$

where $h_1(x), h_2(x)$ are (independent) admissible variations satisfying (necessarily)

$$h_1(a) = h_2(a) = 0$$

$$h_1(b) = h_2(b) = 0$$

Given

$$F(\varepsilon) = \int_a^b L(x, \bar{y}_1 + \varepsilon h_1, \bar{y}_2 + \varepsilon h_2, \bar{y}_1' + \varepsilon h_1', \bar{y}_2' + \varepsilon h_2')\, dx$$

we can compute the first variation $F'(0) = \delta J$

$$F'(0) = \delta J(\bar{y}_1, \bar{y}_2, h_1, h_2)$$

Using shorthand notation

$$F'(\varepsilon) = \int_a^b L_{y_1} h_1 + L_{y_2} h_2 + L_{y_1'} h_1' + L_{y_2'} h_2 \; dx$$

integrating by parts, using the B.C. for $h_k(x)$ and then setting $\varepsilon = 0$

$$F'(0) = \int_a^b \Big(L_{y_1} - \frac{d}{dx} L_{y_1'}\Big) h_1 + \Big(L_{y_2} - \frac{d}{dx} L_{y_2'}\Big) h_2 \, dx$$

each must vanish independently, i.e. if $h_2 \equiv 0$ then first term must vanish if $h_1 \in A_1^*$ arbitrary. Then, since first term vanishes the 2nd must for all $h_2 \in A_2^*$ arbitrary.

The necessary condition that $(\bar{y}_1, \bar{y}_2)$ be an extrema pair yields two differential equations that are coupled.

Theorem A necessary condition $(\bar{y}_1, \bar{y}_2)$ is an extrema for

$$J(y_1, y_2) \equiv \int_a^b L(x, y_1, y_2, y_1', y_2')\,dx$$

with A_k defined before is that they satisfy the coupled Euler Lagrange Eqns

$$(1) \qquad L_{y_1} = \frac{d}{dx} L_{y_1'}$$

$$(2) \qquad L_{y_2} = \frac{d}{dx} L_{y_2'}$$

Euler Lagrange Equations

EXAMPLE Find the extrema of

$$J(y_1, y_2) \equiv \int_0^{\pi/2} (y_1 y_2 - y_1' y_2')\,dx$$

where y_1 and y_2 satisfy the boundary conds:

$$y_1(0) = 0 \qquad y_1(\tfrac{\pi}{2}) = 1$$

$$y_2(0) = 1 \qquad y_2(\tfrac{\pi}{2}) = 0$$

Here

$$L(y_1, y_2, y_1', y_2') = y_1 y_2 - y_1' y_2'$$

Euler Lagrange Eqns

$$L_{y_1} = y_2 \qquad L_{y_1'} = -y_2'$$

$$L_{y_2} = y_1 \qquad L_{y_2'} = -y_1'$$

hence (1)-(2) become

$$(3) \qquad y_2 = -y_2''$$

$$(4) \qquad y_1 = -y_1''$$

and

$$y_1(x) = A_1 \cos x + B_1 \sin x$$

$$y_2(x) = A_2 \cos x + B_2 \sin x$$

Considering the boundary conditions for $y_K(x)$

$$y_1(0) = A_1 = 0$$

$$y_1(\tfrac{\pi}{2}) = B_1 = 1$$

$$y_2(0) = A_2 = 1$$

$$y_2(\tfrac{\pi}{2}) = B_2 = 0$$

and the extrema are, therefore,

$$\bar{y}_1(x) = \sin x \qquad \bar{y}_2(x) = \cos x$$

EXAMPLE Generally if J depends on y_1, y_2, y_1' and y_2' the EL eqns are 4rth order. To see this consider the following example

$$J(y_1, y_2) = \int_0^1 L(y_1, y_2, y_1', y_2')\, dx$$

where

$$L = a y_1^2 + 2b y_1 y_2 + c y_2^2 + 2 y_1' y_2'$$

The Euler Lagrange eqns:

$$L_{y_1} = \frac{d}{dx} L_{y_1'}$$

$$L_{y_2} = \frac{d}{dx} L_{y_2'}$$

are (then) -(after dividing by 2)

$$(1) \quad a y_1 + b y_2 = y_2''$$

$$(2) \quad b y_1 + c y_2 = y_1''$$

This is a fourth order linear system. There are many ways to solve it. One involves eliminating one of the dependent variables.

Repeating the eqns

$$\left.\begin{aligned} &(1) \quad y_2'' = a y_1 + b y_2 \\ &(2) \quad y_1'' = b y_1 + c y_2 \end{aligned}\right\} \text{EL eqns.}$$

Differentiating (2) in x twice

$$\begin{aligned} y_1^{(4)} &= b y_1'' + c y_2'' \quad \text{using (1)} \\ &= b y_1'' + c\,(a y_1 + b y_2) \\ &= b y_1'' + ac y_1 + bc\, y_2 \end{aligned}$$

Now from (2) one can solve for y_2 to use in above

$$y_1^{(4)} = b y_1'' + ac y_1 + b\cancel{c} \cdot \frac{1}{\cancel{c}}(y_1'' - b y_1)$$

$$y_1^{(4)} = 2b y_1'' + (ac - b^2) y_1$$

Thus extrema $(\bar{y}_1, \bar{y}_2)$ must have a $\bar{y}_1(x)$ that solves

$$(3) \qquad y_1^{(4)} - 2b y_1'' + (b^2 - ac) y_1 = 0$$

Such solns have the form $y = Ae^{\lambda x}$ for some λ.

Once (3) is solved, $\bar{y}_2(x)$ is found by solving (2) for $\bar{y}_2$ using your soln $\bar{y}_1(x)$.

Since the general soln of (3) involves four constants the is generally (not always) a unique soln with the four BC deteiming the four constants.

Just a flavor for the difficulties.